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Geometric Parter–Wiener, etc. theory ☆

Charles R. Johnson [a], Carlos M. Saiago [b,*]

[a] Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA

[b] Centro de Matemática e Aplicações (CMA/FCT/UNL), Departamento de Matemática, Faculdade de Ciências e Tecnologia da Universidade NOVA de Lisboa, 2829-516 Quinta da Torre, Portugal

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For real symmetric, or complex Hermitian, matrices whose graph is a tree, there is a well-developed (via several papers) theory about the possible multiplicities of the eigenvalues. It includes a theory of vertices whose removal increases a multiplicity (the "Parter–Wiener, etc. theory" and the "downer branch mechanism"), how to determine maximum multiplicity, lower bounds for the minimum number of distinct eigenvalues, etc. Remarkably, a great deal of this theory may be generalized to geometric multiplicities of general matrices over a field (with very different proofs). We show here what parts of this theory generalize, and, in the process, review the theory.

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* Corresponding author.
E-mail addresses: crjohn@wm.edu (C.R. Johnson), cls@fct.unl.pt (C.M. Saiago).

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1. Introduction

Given a real symmetric, or complex Hermitian, matrix whose graph is a tree T, a remarkable theory has developed for the multiplicities of the eigenvalues. This began with the work [7], was refined in [9] and rather fully explained in [5], as well as references addressing allied issues such as maximum multiplicity [2] and minimum number of distinct eigenvalues [3], etc.

After briefly reviewing this theory, our purpose here is to show that it largely generalizes to general matrices, over a field, whose graph is also a tree when "multiplicity" is replaced by "geometric multiplicity". This substantial generalization requires key algebraic parts of proofs be very different, though certain combinatorial parts remain similar. The essential lesson is that much of multiplicity theory is fundamentally combinatorial and independent of the field or other matrix structure.

There appears to be little interesting to say in the case of algebraic multiplicity, except that almost anything can happen; for example, a n-by-n tridiagonal matrix may have one eigenvalue, with algebraic multiplicity n.

We consider only square, combinatorially symmetric matrices: $A = (a_{ij})$ with $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. If the matrix is n-by-n, the off-diagonal nonzero pattern may be described with an undirected graph $\mathrm{G}(A)$ on n vertices in which $\{i, j\}$ is an edge if and only if $a_{ij} \neq 0$. Given a graph G on n vertices and a field $\mathbb{F}$, denote by $\mathcal{F}(G)$ the set of all $A \in \mathcal{M}_n(\mathbb{F})$ for which $\mathrm{G}(A) = G$. We focus upon the case in which G is a tree T, where the classical "Parter–Wiener, etc." theory is valid. For $A \in \mathcal{F}(G)$, let $\mathrm{gm}_A(\lambda)$ denote the geometric multiplicity of the eigenvalue λ in A; by "eigenvalue" we simply mean a root of the characteristic polynomial, which may lie in an extension field of $\mathbb{F}$, and we also allow $\mathrm{gm}_A(\lambda) = 0$, when λ is not an eigenvalue. The algebraic multiplicity is denoted $\mathrm{am}_A(\lambda)$, and when the two are the same, as in the real symmetric or diagonalizable case, we just use $\mathrm{m}_A(\lambda)$. We also denote the spectrum of A by $\sigma(A)$.

Given a graph G on n vertices and $A \in \mathcal{F}(G)$, if α is an index subset of $\{1, \ldots, n\}$ then $A(\alpha)$ (resp. $G - \alpha$) denotes the principal submatrix of A (resp. induced subgraph of G) resulting from deletion of the rows and columns (resp. vertices) indexed by α. $A[\alpha]$ (resp. $G[\alpha]$) denotes the principal submatrix (resp. induced subgraph) resulting from keeping only the rows and columns (resp. vertices) indexed by α. If $G' = G[\alpha]$ we also often write $A[G']$, meaning the principal submatrix $A[\alpha]$. We abbreviate $A(\{i\})$ (resp. $G - \{i\}$) by $A(i)$ (resp. $G - i$). When G is a tree, $A(i)$ is then a direct sum, whose summands correspond to components of $G - i$, which we call *branches* of G at i.

Lemma 1. *Let G be a graph, v a vertex of G, $A \in \mathcal{F}(G)$ and $\lambda \in \mathbb{F}$. Then, in $A(v)$ there are 3 possibilities, the third occurring only in case $\mathrm{gm}_A(\lambda) \geq 1$:*

1. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$, *which occurs if and only if*

$$\mathrm{rank}\,(A(v) - \lambda I) = \mathrm{rank}\,(A - \lambda I) - 2;$$