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Condition numbers for the tensor rank decomposition

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ABSTRACT

The tensor rank decomposition problem consists of recovering the parameters of the model from an identifiable low-rank tensor. These parameters are analyzed and interpreted in many applications. As tensors are often perturbed by measurement errors in practice, one must investigate to what extent the unique parameters change in order to preserve the validity of the analysis. The magnitude of this change can be bounded asymptotically by the product of the condition number and the norm of the perturbation to the tensor. This paper introduces a condition number that admits a closed expression as the inverse of a particular singular value of Terracini's matrix, which represents the tangent space to the set of tensors of fixed rank. A practical algorithm for computing this condition number is presented. The latter's elementary properties such as scaling and orthogonal invariance are established. Rank-1 tensors are always well-conditioned. The class of weak 3-orthogonal tensors, which includes orthogonally decomposable tensors, contains both well-conditioned and ill-conditioned problems. Numerical experiments confirm that the condition number yields a good estimate of the magnitude of the change of the parameters when the tensor is perturbed.

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1. Introduction

A *tensor* is an element of $\mathrm{T} = \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, $n_k \in \mathbb{N}$, and $\otimes$ denotes the tensor product. A *rank*-1 *tensor* in T is defined as $\mathbf{a}^1 \otimes \cdots \otimes \mathbf{a}^d$ with $\mathbf{a}^k \in \mathbb{F}^{n_k} \setminus \{0\}$. Every tensor is expressible as a linear combination of rank-1 tensors:

$$\mathfrak{A} = \sum_{i=1}^{r} \mathbf{a}_i^1 \otimes \mathbf{a}_i^2 \otimes \cdots \otimes \mathbf{a}_i^d, \quad \text{where } \mathbf{a}_i^k \in \mathbb{F}^{n_k}. \tag{1}$$

If the number of terms r is minimal, then this decomposition due to Hitchcock [29] is called a *tensor rank decomposition*. The integer r is then called the *rank* of the tensor.

The *representative* $\mathfrak{p} = (\mathbf{a}^1, \ldots, \mathbf{a}^d)$ of a rank-1 tensor $\mathbf{a}^1 \otimes \cdots \otimes \mathbf{a}^d$ is called *essentially unique* or *identifiable* because the vectors $\mathbf{a}^k$ are unique up to scaling: if $\mathfrak{q} = (\mathbf{b}^1, \ldots, \mathbf{b}^d)$ is such that $\mathbf{a}^1 \otimes \cdots \otimes \mathbf{a}^d = \mathbf{b}^1 \otimes \cdots \otimes \mathbf{b}^d$, then $\mathbf{b}^k = \lambda_k \mathbf{a}^k$ for $k = 1, \ldots, d$, and furthermore $\lambda_1 \cdots \lambda_d = 1$. A rank-$r$ tensor with $d \geq 3$ often admits an r-*identifiable* decomposition [16,23,34]: the set of rank-1 tensors appearing in a decomposition is uniquely determined. For instance, if $d = 3$ the well-known criterion due to Kruskal [34] states that (1) is the essentially unique decomposition if $r \leq \frac{1}{2}(k_1 + k_2 + k_3 - 2)$, where k_i is the *Kruskal rank* of $A_i = \{\mathbf{a}_j^i\}_{j=1}^r$; that is, the integer k_i such that every subset of A_i of cardinality k_i forms a linearly independent set.

Identifiability of a tensor rank decomposition is essential in several applications, such as in the parameter identification problem in latent variable models [3,4]. Models whose parameters can be inferred with this decomposition and symmetrical variants were recently surveyed in a unified tensor-based framework [4]; they include exchangeable single topic models, hidden Markov models, and Gaussian mixture models. The blind source separation problem, or independent component analysis, in signal processing is another example where the uniqueness of the symmetric tensor rank decomposition of a cumulant tensor is key in identifying the statistically independent signals [18].

In all of the above applications, a mathematical theory guarantees that the tensor $\mathfrak{A}$ has a rank-r decomposition; however, in practice this tensor is often estimated from imperfect measurements. This entails that the true but unknown tensor $\mathfrak{A}$ is approximated by the estimated tensor $\mathfrak{A}''$ whose rank, in general, is not equal to r. Therefore, it is common practice to approximate $\mathfrak{A}''$ by a nearby rank-r tensor $\mathfrak{A}'$. The latter's rank-r decomposition then serves as a proxy for $\mathfrak{A}$'s decomposition. This is a standard approach in data-analytic applications, yet one primordial question has thus far remained unanswered:

> What is the relationship between the essentially unique decompositions of two rank-r tensors $\mathfrak{A}$ and $\mathfrak{A}'$ when $\|\mathfrak{A} - \mathfrak{A}'\| \leq \epsilon$ for small perturbations $\epsilon > 0$?

The goal of this paper is characterizing that relationship to first order via a study of the *condition number* of the associated computational problem of computing a tensor rank decomposition.