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Nullities of graphs with given order, matching number and cyclomatic number revisited

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A B S T R A C T

For a (simple) graph G, we denote by $|V(G)|, |E(G)|, \eta(G)$ and $m(G)$ respectively the order, the number of edges, the nullity, and the matching number of G. It was shown by Wang and Wong (2014) that for every graph G, $|V(G)| - 2m(G) - c(G) \leq \eta(G) \leq |V(G)| - 2m(G) + 2c(G)$, where $c(G) := |E(G)| - |V(G)| + \theta(G)$ is the cyclomatic number of G, $\theta(G)$ being the number of components of G. Graphs with the maximal nullity condition have been characterized by Song et al. (2015), and graphs with the minimal nullity condition have also been characterized independently by Rula et al. (2016) and Wang (2016). Earlier Guo et al. (2009) had also shown that for a unicyclic graph G, $\eta(G) - |V(G)| + 2m(G)$ can take only one of the values $-1, 0$ or 2, and they characterized these three types of unicyclic graphs. In this paper, exploiting the concepts of canonical star associated with a rooted tree, the canonical unicyclic graph associated with a unicyclic graph and a crucial subgraph of a graph, we correct, complete and extend the work of previous authors on this topic. We give a nontrivial proof for the fact that the crucial subgraphs of a graph, as introduced by Wang (2016), are unique up to isomorphism. More complete lists of characterizations for the three types of unicyclic graphs, for nonsingular unicyclic graphs, and for graphs with the minimal or maximal nullity conditions are found. It is proved that if c, n are given positive integers with $n \geq 6c+2$, then for any integer

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$k, -c \leq k \leq 2c, k \neq 2c-1$, there exists a connected graph G of order n that satisfies $c(G) = c$ and $\eta(G)-|V(G)|+2m(G) = k$; however, if k is replaced by $2c-1$ then there is no graph G of any order that satisfies the corresponding conditions.

1. Introduction

Recently, Wang and Wong [12, Theorem 1.1(i)] obtained bounds for the matching number of a graph G in terms of its rank and cyclomatic number. These bounds can be rewritten in an equivalent form as bounds for the nullity $\eta(G)$ of G, namely, $|V(G)|-2m(G)-c(G) \leq \eta(G) \leq |V(G)|-2m(G)+2c(G)$, where $m(G)$ denotes the matching number and $c(G)$ the cyclomatic number, i.e., $c(G) = |E(G)| - |V(G)| + \theta(G), \theta(G)$ being the number of (connected) components of G. Graphs G with the maximal nullity condition (i.e., for which $\eta(G)$ attain the upper bound) have been characterized by Song et al. [9], and graphs G with the minimal nullity condition have also been characterized independently by Rula et al. [8] and Wang [11]. Earlier Guo et al. [5] had made an intensive study on the nullity of unicyclic graphs. They found that if G is unicyclic then $\eta(G)$ can take only one of the three values $|V(G)| - 2m(G) - 1, |V(G)| - 2m(G)$ or $|V(G)| - 2m(G) + 2$, and they characterized these three types of unicyclic graphs.

However, these previous works appear to be incomplete and contain mathematical errors. In particular, the concepts of the canonical unicyclic graph associated with a unicyclic graph and that of a crucial subgraph of a graph were introduced in [5] and [11] respectively, but the uniqueness issue of these mathematical objects were not treated and the concepts were hardly developed. This work was initiated by our attempt to fill the gaps, lay the foundation, and complete the work on this topic.

Now we describe the contents of this paper in some detail.

In Section 2, we give some necessary definitions and preparatory results. By introducing the concept of the canonical star associated with a rooted tree, we establish the uniqueness of the canonical unicyclic graph associated with a unicyclic graph. We provide a nontrivial proof for the fact that the crucial subgraph of a graph is unique up to graph isomorphism. In connection with nullities, matching numbers or perfect matchings, some basic results involving G, G' and $G - G'$, where G' is a crucial subgraph of G, are also given.

In Section 3, we derive a strengthened version of the characterizations of the three types of unicyclic graphs as given in [5, Theorems 2.1 and Corollary 2.1]. Unlike in [5], our proofs are more conceptual and do not rely on the use of the Sachs theorem.