Linear Algebra and its Applications 535 (2017) 160–170

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Eigenvalue multiplicity in quartic graphs

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A R T I C L E I N F O

Article history:
Received 23 November 2016
Accepted 30 August 2017
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Eigenvalue
Quartic graph
Star complement

A B S T R A C T

Let G be a connected quartic graph of order n with μ as an eigenvalue of multiplicity k. We show that if $\mu \notin \{-1, 0\}$ then $k \leq (2n-5)/3$ when $n \leq 22$, and $k \leq (3n-1)/5$ when $n \geq 23$. If $\mu \in \{-1, 0\}$ then $k \leq (2n+2)/3$, with equality if and only if $G = K_5$ (with $\mu = -1$) or $G = K_{4,4}$ (with $\mu = 0$).

1. Introduction

Let G be a regular graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k and codimension t. From [1, Theorem 3.1], we know that if $\mu \notin \{-1, 0\}$ then $k \leq n - \frac{1}{2}(-1 + \sqrt{8n+9})$, equivalently $k \leq \frac{1}{2}(t+1)(t-2)$. For connected quartic graphs, a bound which is linear in t follows easily from the equation $\mathrm{tr}(A) = 0$. To see this, we suppose that $k \geq \frac{1}{2}n$, i.e. $k \geq t$. Then G is non-bipartite; also μ is an integer,

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http://dx.doi.org/10.1016/j.laa.2017.08.023

for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity k. It follows that if G is a connected quartic graph then $\mu \in \{-3,-2,1,2,3\}$ (see [6, Sections 1.3 and 3.2]). Let d be the mean of the eigenvalues other than 4 and μ, so that $4+k\mu+(n-k-1)d=0$. We have $-4<d<4$, and so:

(a) if $\mu=-3$ then $k<\frac{4}{7}n$, i.e. $k<\frac{4}{3}t$;
(b) if $\mu=-2$ then $k<\frac{2}{3}n$, i.e. $k<2t$;
(c) if $\mu=1$ then $k<\frac{4}{5}n-\frac{8}{5}$, i.e. $k<4t-8$;
(d) if $\mu=2$ then $k<\frac{2}{3}n-\frac{4}{3}$, i.e. $k<2t-4$;
(e) if $\mu=3$ then $k<\frac{4}{7}n-\frac{8}{7}$, i.e. $k<\frac{4}{3}t-\frac{8}{3}$.

We show first that $k\le 2t-5$ whenever $\mu\notin\{-1,0\}$. Then k is at most $\lfloor(2n-5)/3\rfloor$, a bound which is sharp for $n=6,9,12$. The arguments are somewhat different from those in the paper [9], where a corresponding bound for cubic graphs was established. Section 2 contains the required results on star complements, while Section 3 provides details of the proof. It is quickly established that the bound holds when $t>9$ or $n>23$, and subsequently we are able to improve the bound to $(3n-1)/5$ when $n\ge 23$. The large number of quartic graphs of order ≤ 23 justifies our case-by-case analysis when $t\le 9$: the cases $n>17$ are relatively easy to deal with, but there are already 86221634 connected quartic graphs of order 17 [8, Sequence A006820]. In Section 4 we show that when $\mu\in\{-1,0\}$ we have $k\le(2n+2)/3$, with equality if and only if $G=K_5$ (with $\mu=-1$) or $G=K_{4,4}$ (with $\mu=0$).

2. Preliminaries

Let G be a graph of order n with μ as an eigenvalue of multiplicity k. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X|=k$ and the induced subgraph $G-X$ does not have μ as an eigenvalue. In this situation, $G-X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [6, Chapter 5]. We shall require the following results where we write $u\sim v$, meaning that vertices u and v are adjacent. For any $U\subseteq V(G)$, we write G_U for the subgraph of G induced by U, and $\Delta_U(v)$ for the set $\{u\in U : u\sim v\}$. For the subgraph H of G it is convenient to write $\Delta_H(v)$ for $\Delta_{V(H)}(v)$.

Theorem 2.1. *(See [6, Theorem 5.1.7]) Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C\end{pmatrix}$, where A_X is the adjacency matrix of G_X.*

(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{1}$$