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Pascal eigenspaces and invariant sequences of the first or second kind ☆

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An infinite real sequence $\{a_n\}$ is called an invariant sequence of the first (resp., second) kind if $a_n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k a_k$ (resp., $a_n = \sum_{k=n}^{\infty} \binom{k}{n}(-1)^k a_k$). We review and investigate invariant sequences of the first and second kind, and study their relationships using similarities of Pascal-type matrices and their eigenspaces.

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1. Introduction

Inverse relations play an important role in combinatorics [11]. The *binomial inversion formula*, which states that for sequences $\{a_n\}$ and $\{b_n\}$ $(n = 0, 1, 2, \dots)$,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k, \tag{1.1}$$

is a typical inverse relation considered in [6,8,10,14–16]. Specifically, (1.1) motivated Sun [14] to investigate the following sequences.

Definition 1.1. Let $\{a_n\}$ $(n = 0, 1, 2, \dots)$ be a sequence such that

$$(-1)^{s-1}\, a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k; \quad s = 1 \ \text{ or } \ s = 2. \tag{1.2}$$

We refer to $\{a_n\}$ as an *invariant sequence* (when $s = 1$) or an *inverse invariant sequence* (when $s = 2$) *of the first kind.*

Several examples of invariant sequences of the first kind can be found in [14], including

$$\{\frac{1}{2^n}\},\ \{nF_{n-1}\},\ \{L_n\},\ \{(-1)^n B_n\} \ \ (n \geq 0),$$

where $F_{-1} = 0$ and $\{F_n\}$, $\{L_n\}$, and $\{B_n\}$ are the Fibonacci sequence, Lucas sequence, and Bernoulli numbers [7], respectively. In this paper, we will establish (see Lemma 2.1) the *modified binomial inversion formula*, that is,

$$a_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k b_k \quad \text{if and only if} \quad b_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k a_k. \tag{1.3}$$

Motivated by (1.3), we will introduce and consider the following sequences.

Definition 1.2. Let $\{a_n\}$ $(n = 0, 1, 2, \dots)$ be a sequence such that

$$(-1)^{s-1} a_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k a_k; \quad s = 1 \ \text{ or } \ s = 2. \tag{1.4}$$

We refer to $\{a_n\}$ as an *invariant sequence* (when $s = 1$) or an *inverse invariant sequence* (when $s = 2$) *of the second kind.*

Naturally arising are the questions of existence, identification, and construction of (inverse) invariant sequences of the second kind, as well as the problem of characterizing such sequences and examining their relationship to their counterparts of the first kind. Invariant sequences, which are also called self-inverse sequences in [15], have indeed been