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A construction of distance cospectral graphs

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ABSTRACT

The distance matrix of a connected graph is the symmetric matrix with columns and rows indexed by the vertices and entries that are the pairwise distances between the corresponding vertices. We give a construction for graphs which differ in their edge counts yet are cospectral with respect to the distance matrix. Further, we identify a subgraph switching behavior which constructs additional distance cospectral graphs. The proofs for both constructions rely on a perturbation of (most of) the distance eigenvectors of one graph to yield the distance eigenvectors of the other.

1. Introduction

Spectral graph theory explores the relationship between a graph and the eigenvalues (i.e., spectrum) of a matrix associated with that graph. There are a handful of common ways to associate a matrix to a graph, and the spectrum of each matrix holds a variety of information about the graph (see [3]). However, each matrix also has limitations in what information its spectrum can contain. This is seen in the existence of *cospectral graphs*,

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or graphs that are fundamentally different yet yield the same spectrum for a particular matrix.

By exploring cospectral graphs, we further our understanding of the limitations of each type of matrix. One of the most well-known constructions of cospectral graphs for the adjacency matrix is Godsil–McKay switching. This is done by defining specific subsets of the vertices of a particular graph and constructing a cospectral mate by exchanging edges and non-edges between these subsets. Godsil and McKay [4] prove the adjacency matrices of two graphs related by this edge switching are similar, and therefore the graphs are cospectral.

In this paper, we consider cospectral graphs for the *distance matrix*. The distance matrix $D^{(G)} = \left[d_{ij}^{(G)}\right]$ of a connected graph $G = (V(G), E(G))$ is a symmetric matrix such that $d_{ij}^{(G)}$ is the distance, or length of the shortest path, between vertices i and j. Its multiset of eigenvalues is the *distance spectrum* of G and two graphs are considered to be distance cospectral if their distance spectra are the same. There has been extensive work done on the distance spectra of graphs (see [2] for a survey of recent results).

However, relatively little is known in regard to distance cospectral pairs. McKay [5] gives a construction for distance cospectral trees by considering any rooted tree and identifying the root with the root of one of two particular trees. Further, he proves the complement graphs of trees constructed in this fashion are also distance cospectral. Both proofs rely on manipulation of the distance characteristic polynomial. While this paper was under review, Abaid et al. gave a construction for distance cospectral graphs by “blowing up” vertices of distance cospectral graphs with cliques and independent sets [1]. We note that in both constructions, cospectral pairs must contain the same number of edges. In particular, prior to this paper it was not known whether a family could be constructed where distance cospectral pairs could have differing numbers of edges.

In this paper, we give a construction for distance cospectral graphs with differing numbers of edges in Section 2, and in Section 3 we describe a local edge switching behavior which produces more distance cospectral graphs. While these distance switching pairs do not differ in number of edges, they do account for all distance cospectral pairs on seven vertices (see Fig. 4). Finally, in Section 4, we consider further questions of interest for the distance matrix. We complete the introduction with an elementary discussion of graph identification, a process which will be used in subsequent sections.

1.1. Graph identification

Throughout our constructions, we frequently make use of *graph identification*. As such, we define it here and state some observations about distances between vertices in graphs formed in this way. Let G, K be graphs and let $u \in V(G)$, $v \in V(K)$. We construct the graph $GK(u, v)$ by identifying the vertices u and v into a new vertex uv in the graph $G \cup K$. When clear context allows, we will denote this graph GK.