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The characteristic polynomial of an algebra and representations

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ABSTRACT

We pose two new linear preserver problems. First, given a field $\mathbf{k}$ and a finite dimensional $\mathbf{k}$-algebra B we show that the only linear maps $\phi\colon \mathbf{k}^{\times d} \to B$ which preserve the unit and the n-th roots of unity (for some $n > 2$ coprime to char(k)) are the algebra homomorphisms. Second, we consider linear maps $\phi\colon A \to B$ between finite dimensional $\mathbf{k}$-algebras which preserve the Cayley–Hamilton relations. We show that if preservation of the Cayley–Hamilton property is understood in a certain non-commutative sense, and $A = \mathbf{k}^{\times d}$, then the only such linear mappings are the algebra homomorphisms.

Suppose that $\mathbf{k}$ is a field and let A be a finite dimensional, associative, unital $\mathbf{k}$-algebra. Often one is interested in studying the finite-dimensional representations of A. Of course, a finite dimensional representation of A is simply a finite dimensional $\mathbf{k}$-vector space M and a $\mathbf{k}$-algebra homomorphism $A \to \mathrm{End}_{\mathbf{k}}(M)$. In this article we will not consider representations of algebras, but rather how to determine if a $\mathbf{k}$-linear map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ is actually a homomorphism. We restrict our attention to the case where A is

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a product of copies of $\mathbf{k}$. If $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ is a representation then certainly, if $a \in A$ satisfies $a^m = 1$ then $\phi(a)^m = \mathrm{id}$ as well. Our first Theorem is a remarkable converse to this elementary observation.

Theorem A. *Let $\mathbf{k}$ be a field of characteristic unequal to 2. Let $A = \mathbf{k}^{\times d}$ and B be a finite dimensional $\mathbf{k}$-algebra. Fix $n > 2$ and suppose that $\mathbf{k}$ contains a full set of n-th roots of unity. If a linear map $\phi\colon A \to B$ satisfies $\phi(1_A) = 1_B$ and $\phi(a)^n = 1_B$ for each $a \in A$ such that $a^n = 1_A$, then ϕ is an algebra homomorphism.*

Remark 1. In the case $A = \mathrm{Mat}_n(\mathbf{k})$ and $k > 1$, there is a complete classification of those linear mappings $\phi\colon A \to \mathrm{Mat}_m(\mathbf{k})$ such that $\phi(X^k) = \phi(X)^k$ and of those mappings where if $X^k = X$ then $\phi(X)^k = \phi(X)$. See [3,1,9] and the references therein. Such mappings need not be homomorphisms (or antihomomorphisms).

Consider the regular representation $\mu_L : A \to \mathrm{End}_{\mathbf{k}}(A)$ of A on itself by left multiplication. For $a \in A$, let $\chi_a(t)$ and $\overline{\chi}_a(t)$ be the characteristic and minimal polynomials of $\mu_L(a)$, respectively. We note that $\chi_a(a) = \overline{\chi}_a(a) = 0$ in A. Therefore if M is a finite dimensional left A module with structure map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ then $\chi_a(\phi(a)) = \overline{\chi}_a(\phi(a)) = 0$ in $\mathrm{End}_{\mathbf{k}}(M)$. The notion of assigning a characteristic polynomial to each element of an algebra and considering representations which are compatible with this assignment has appeared in [8]. This idea has been applied to some problems in noncommutative geometry as well [5].

Definition 2. Suppose that $\phi : A \to B$ is a $\mathbf{k}$-linear map, where B is a $\mathbf{k}$-algebra. We say that ϕ is a *characteristic morphism* if $\chi_a(\phi(a)) = 0$ for all $a \in A$. We say that ϕ is *minimal-characteristic* if, moreover, $\overline{\chi}_a(\phi(a)) = 0$ for all $a \in A$.

Remark 3. While the notion of characteristic morphism appears to be new, especially in the case where A and B are general $\mathbf{k}$-algebras, several related notions have been studied. Suppose that $A = \mathrm{Mat}_n(\mathbf{k})$ and ϕ is a linear endomorphism of A. If ϕ preserves the determinant then, according to a result of Frobenius [4], $\phi(X) = MXN$ or $\phi(X) = MX^TN$ where $\det(MN) = 1$. Furthermore, Marcus and Purves [6] extend this by proving that if ϕ preserves any one of the coefficients of the characteristic polynomial (other than the $(n-1)$-st, $(n-2)$-nd, $(n-3)$-rd or 0-th) then either $\phi(X) = MXM^{-1}$ or $\phi(X) = MX^TM^{-1}$ (up to multiplying by an appropriate root of unity). Their result implies that a characteristic endomorphism of $A = \mathrm{Mat}_n(\mathbf{k})$ is either an automorphism or anti-automorphism.

It is natural to ask whether or not the notions of characteristic morphism and minimal-characteristic morphism are weaker than the notion of algebra morphism. Let us address minimal-characteristic morphisms first.