

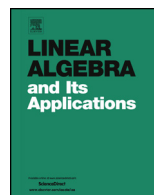


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## Linear Algebra and its Applications

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## The characteristic polynomial of an algebra and representations

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## ABSTRACT

We pose two new linear preserver problems. First, given a field  $\mathbf{k}$  and a finite dimensional  $\mathbf{k}$ -algebra  $B$  we show that the only linear maps  $\phi: \mathbf{k}^{\times d} \rightarrow B$  which preserve the unit and the  $n$ -th roots of unity (for some  $n > 2$  coprime to  $\text{char}(\mathbf{k})$ ) are the algebra homomorphisms. Second, we consider linear maps  $\phi: A \rightarrow B$  between finite dimensional  $\mathbf{k}$ -algebras which preserve the Cayley–Hamilton relations. We show that if preservation of the Cayley–Hamilton property is understood in a certain non-commutative sense, and  $A = \mathbf{k}^{\times d}$ , then the only such linear mappings are the algebra homomorphisms.

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Suppose that  $\mathbf{k}$  is a field and let  $A$  be a finite dimensional, associative, unital  $\mathbf{k}$ -algebra. Often one is interested in studying the finite-dimensional representations of  $A$ . Of course, a finite dimensional representation of  $A$  is simply a finite dimensional  $\mathbf{k}$ -vector space  $M$  and a  $\mathbf{k}$ -algebra homomorphism  $A \rightarrow \text{End}_{\mathbf{k}}(M)$ . In this article we will not consider representations of algebras, but rather how to determine if a  $\mathbf{k}$ -linear map  $\phi: A \rightarrow \text{End}_{\mathbf{k}}(M)$  is actually a homomorphism. We restrict our attention to the case where  $A$  is

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a product of copies of  $\mathbf{k}$ . If  $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$  is a representation then certainly, if  $a \in A$  satisfies  $a^m = 1$  then  $\phi(a)^m = \text{id}$  as well. Our first Theorem is a remarkable converse to this elementary observation.

**Theorem A.** *Let  $\mathbf{k}$  be a field of characteristic unequal to 2. Let  $A = \mathbf{k}^{\times d}$  and  $B$  be a finite dimensional  $\mathbf{k}$ -algebra. Fix  $n > 2$  and suppose that  $\mathbf{k}$  contains a full set of  $n$ -th roots of unity. If a linear map  $\phi : A \rightarrow B$  satisfies  $\phi(1_A) = 1_B$  and  $\phi(a)^n = 1_B$  for each  $a \in A$  such that  $a^n = 1_A$ , then  $\phi$  is an algebra homomorphism.*

**Remark 1.** In the case  $A = \text{Mat}_n(\mathbf{k})$  and  $k > 1$ , there is a complete classification of those linear mappings  $\phi : A \rightarrow \text{Mat}_m(\mathbf{k})$  such that  $\phi(X^k) = \phi(X)^k$  and of those mappings where if  $X^k = X$  then  $\phi(X)^k = \phi(X)$ . See [3,1,9] and the references therein. Such mappings need not be homomorphisms (or antihomomorphisms).

Consider the regular representation  $\mu_L : A \rightarrow \text{End}_{\mathbf{k}}(A)$  of  $A$  on itself by left multiplication. For  $a \in A$ , let  $\chi_a(t)$  and  $\bar{\chi}_a(t)$  be the characteristic and minimal polynomials of  $\mu_L(a)$ , respectively. We note that  $\chi_a(a) = \bar{\chi}_a(a) = 0$  in  $A$ . Therefore if  $M$  is a finite dimensional left  $A$  module with structure map  $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$  then  $\chi_a(\phi(a)) = \bar{\chi}_a(\phi(a)) = 0$  in  $\text{End}_{\mathbf{k}}(M)$ . The notion of assigning a characteristic polynomial to each element of an algebra and considering representations which are compatible with this assignment has appeared in [8]. This idea has been applied to some problems in noncommutative geometry as well [5].

**Definition 2.** Suppose that  $\phi : A \rightarrow B$  is a  $\mathbf{k}$ -linear map, where  $B$  is a  $\mathbf{k}$ -algebra. We say that  $\phi$  is a *characteristic morphism* if  $\chi_a(\phi(a)) = 0$  for all  $a \in A$ . We say that  $\phi$  is *minimal-characteristic* if, moreover,  $\bar{\chi}_a(\phi(a)) = 0$  for all  $a \in A$ .

**Remark 3.** While the notion of characteristic morphism appears to be new, especially in the case where  $A$  and  $B$  are general  $\mathbf{k}$ -algebras, several related notions have been studied. Suppose that  $A = \text{Mat}_n(\mathbf{k})$  and  $\phi$  is a linear endomorphism of  $A$ . If  $\phi$  preserves the determinant then, according to a result of Frobenius [4],  $\phi(X) = MXN$  or  $\phi(X) = MX^T N$  where  $\det(MN) = 1$ . Furthermore, Marcus and Purves [6] extend this by proving that if  $\phi$  preserves any one of the coefficients of the characteristic polynomial (other than the  $(n - 1)$ -st,  $(n - 2)$ -nd,  $(n - 3)$ -rd or 0-th) then either  $\phi(X) = MXM^{-1}$  or  $\phi(X) = MX^T M^{-1}$  (up to multiplying by an appropriate root of unity). Their result implies that a characteristic endomorphism of  $A = \text{Mat}_n(\mathbf{k})$  is either an automorphism or anti-automorphism.

It is natural to ask whether or not the notions of characteristic morphism and minimal-characteristic morphism are weaker than the notion of algebra morphism. Let us address minimal-characteristic morphisms first.

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