Linear Algebra and its Applications 530 (2017) 202–219

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Pseudo-direct sums and wreath products of loose-coherent algebras with applications to coherent configurations

Bangteng Xu

Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, KY 40475, USA

ARTICLE INFO

Article history:
Received 21 March 2017
Accepted 3 May 2017
Available online 8 May 2017
Submitted by R. Brualdi

In memory of Hongyou Wu

MSC:
16S50
05E30

Keywords:
Loose-coherent algebras
Pseudo-direct sums
Wreath products
Coherent configurations
Terwilliger algebras

ABSTRACT

We introduce the notion of a loose-coherent algebra, which is a special semisimple subalgebra of the matrix algebra, and define two operations to obtain new loose-coherent algebras from the old ones: the pseudo-direct sum and the wreath product. For two arbitrary coherent configurations $\mathfrak{C}$, $\mathfrak{D}$ and their wreath product $\mathfrak{C} \wr \mathfrak{D}$, it is difficult to express the Terwilliger algebra $\mathcal{T}_{(x,y)}(\mathfrak{C} \wr \mathfrak{D})$ in terms of the Terwilliger algebras $\mathcal{T}_x(\mathfrak{C})$ and $\mathcal{T}_y(\mathfrak{D})$. By using the concept and operations of loose-coherent algebras, we find a very simple such expression. As a direct consequence of this expression, we obtain the central primitive idempotents of $\mathcal{T}_{(x,y)}(\mathfrak{C} \wr \mathfrak{D})$ in terms of the central primitive idempotents of $\mathcal{T}_x(\mathfrak{C})$ and $\mathcal{T}_y(\mathfrak{D})$. Many results in [4,6,10,12] are special cases of the results in this paper.

E-mail address: bangteng.xu@eku.edu.

http://dx.doi.org/10.1016/j.laa.2017.05.009

1. Introduction

As an analogue of a point stabilizer of a transitive permutation group, Terwilliger [13] introduced the so-called subconstituent algebra for a commutative association scheme. This is a (noncommutative) semisimple subalgebra of the matrix algebra of all $n \times n$ matrices over the complex field, where n is the order of the association scheme. The subconstituent algebra is more often called a Terwilliger algebra (or T-algebra) nowadays. Many researchers have studied the T-algebra and its applications (for example, see [1–4,6,8,10–12,14] and the references therein). In general, it is very difficult to describe the structure of a T-algebra and its non-principal primitive ideals. The wreath product of association schemes is an important tool to construct new association schemes from the old ones. Several papers have investigated the central primitive idempotents of the T-algebra of the wreath product of two association schemes (cf. [2,4,6,10]). The computations employed in these works are rather intricate. Association schemes are a special class of coherent configurations. The T-algebras are also defined for any coherent configurations. For two arbitrary coherent configurations $\mathfrak{C}$, $\mathfrak{D}$ and their wreath product $\mathfrak{C} \wr \mathfrak{D}$, a more general and difficult problem is how to express the Terwilliger algebra $\mathcal{T}_{(x,y)}(\mathfrak{C} \wr \mathfrak{D})$ in terms of the Terwilliger algebras $\mathcal{T}_x(\mathfrak{C})$ and $\mathcal{T}_y(\mathfrak{D})$.

Our work has been inspired by [10,12]. Let $\mathcal{S}$ and $\mathcal{R}$ be two arbitrary association schemes, and let $\mathcal{S} \wr \mathcal{R}$ be the wreath product of them. By decomposing the T-algebra of $\mathcal{S} \wr \mathcal{R}$ into the sum of some subalgebras (see [10, Theorem 4.1]), we obtain the central primitive idempotents of the T-algebra of $\mathcal{S} \wr \mathcal{R}$ (see [10, Theorem 5.3]). The decomposition in [10, Theorem 4.1] does not give an explanation of the structure of the T-algebra of $\mathcal{S} \wr \mathcal{R}$, and its proof is very difficult and complicated. In [12] we introduce the concept of the combinatorial extension of a T-algebra by a coherent algebra, and prove that the T-algebra of $\mathcal{S} \wr \mathcal{R}$ is isomorphic to the combinatorial extension of the T-algebra of $\mathcal{S}$ by a coherent algebra under the condition that the T-algebra of $\mathcal{R}$ is a coherent algebra (see [12, Theorem 1.4]). There are many association schemes whose T-algebras are coherent algebras; but in general, a T-algebra is not a coherent algebra. So [12, Theorem 1.4] only describes the structures of the T-algebra of $\mathcal{S} \wr \mathcal{R}$ for those $\mathcal{R}$ whose T-algebras are coherent algebras.

In this paper we will first introduce the concept of a loose-coherent algebra (see Definition 1.1 below). Then we define two operations for the loose-coherent algebras: the pseudo-direct sum (see Definition 1.3) and the wreath product (see Definition 1.5). We will prove that the pseudo-direct sum and wreath product of two loose-coherent algebras are also loose-coherent algebras (see Theorems 1.4 and 1.6). For two arbitrary coherent configurations $\mathfrak{C}$, $\mathfrak{D}$ and their wreath product $\mathfrak{C} \wr \mathfrak{D}$, by using the pseudo-direct sum and wreath product of loose-coherent algebras, we will express the Terwilliger algebra $\mathcal{T}_{(x,y)}(\mathfrak{C} \wr \mathfrak{D})$ in terms of the Terwilliger algebras $\mathcal{T}_x(\mathfrak{C})$ and $\mathcal{T}_y(\mathfrak{D})$ (see Theorem 1.7). Furthermore, as a direct consequence of Theorem 1.7, we obtain the central primitive idempotents of $\mathcal{T}_{(x,y)}(\mathfrak{C} \wr \mathfrak{D})$ in terms of the central primitive idempotents of $\mathcal{T}_x(\mathfrak{C})$ and