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Linear mappings preserving the diameter of the numerical range

Jor-Ting Chan, Kong Chan

Department of Mathematics, The University of Hong Kong, Hong Kong

A R T I C L E I N F O

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A B S T R A C T

Let H be a complex Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. For every $A \in B(H)$, the numerical range of A is the set $W(A) = \{\langle Ax, x\rangle : x \in H \text{ and } \|x\| = 1\}$ and the diameter of $W(A)$ is the number $d(W(A)) = \sup\{|\lambda - \mu| : \lambda, \mu \in W(A)\}$. A mapping $T : B(H) \to B(H)$ is called a diameter preserver if $d(W(T(A))) = d(W(A))$ for all $A \in B(H)$. In this article we give a characterization of surjective linear diameter preservers.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators on H. For each $A \in B(H)$, the numerical range and numerical radius of A are defined respectively by

$$W(A) = \{\langle Ax, x\rangle : x \in H \text{ and } \|x\| = 1\} \quad \text{and} \quad r(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

E-mail address: jtchan@hku.hk (J.-T. Chan).

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The *diameter* of $W(A)$ is defined as usual by

$$d(W(A)) = \sup\{|\lambda - \mu| : \lambda, \mu \in W(A)\}.$$

We shall call this number the diameter of A and write $d(A)$ instead of $d(W(A))$. A mapping $T : B(H) \to B(H)$ is called a *diameter preserver* if it preserves the diameter of A for each $A \in B(H)$, i.e., if

$$d(T(A)) = d(A) \quad \text{for all } A \in B(H).$$

The main result of this article is a characterization of surjective linear diameter preservers.

Our study is inspired by [5,2] and [4], in which the authors studied diameter preservers on $C(X)$, the Banach space of all continuous functions on a compact Hausdorff space X. Here the diameter of an $f \in C(X)$ is the diameter of its range, i.e.,

$$d(f) = \sup\{|f(x) - f(y)| : x, y \in X\}.$$

The result has been extended in different directions. For example, in [3] and [1], the authors considered diameter preservers on $A(S)$, the Banach space of continuous affine functions on a compact convex set S. This is related to our study as follows. Let S be the *state-space* of $B(H)$, i.e.,

$$S = \{\phi \in B(H)^* : \phi(I) = 1 = \|\phi\|\}.$$

The set S is convex and compact in the weak* topology. Every $A \in B(H)$ can be identified with the continuous affine function $\phi \mapsto \phi(A)$ on S. The sup-norm and the diameter of the function are the numerical radius and the diameter of A respectively. Our study can be viewed as the study of diameter preservers on $A(S)$. This consideration enables us to make use of the tools developed in these papers.

Another related problem is the following. For any $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, the c-numerical range and c-numerical radius are defined respectively by

$$W_c(A) = \left\{ \sum_{j=1}^{n} c_j \langle Ax_j, x_j \rangle : \{x_1, \ldots, x_j\} \text{ is an orthonormal set in } H \right\}$$

and $r_c(A) = \sup\{|\lambda| : \lambda \in W_c(A)\}$. By [12, Lemma 1], the diameter of A is equal to $r_c(A)$ for $c = (1, -1)$. When $\dim H < \infty$, complete descriptions of c-numerical radius preservers (for general c, not just $(1, -1)$) have been given by Li and Tsing in [9] and [10]. In other words, the problem (indeed a more general one) is solved when $\dim H < \infty$. In the sequel, we always assume that H is infinite-dimensional.

Note that $d(\cdot)$ is a semi-norm on $B(H)$. It is clearly not positive definite. Indeed $d(A) = 0$ if and only if $W(A)$ is a singleton set if and only if $A = \lambda I$ for some $\lambda \in \mathbb{C}$,