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# Derivations of tensor products of nonassociative algebras

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ABSTRACT

Let $R$ and $S$ be nonassociative unital algebras. Assuming that either one of them is finite dimensional or both are finitely generated, we show that every derivation of $R \otimes S$ is the sum of derivations of the following three types: (a) $\operatorname{ad} u$ where $u$ belongs to the nucleus of $R \otimes S$, (b) $L_z \otimes f$ where $f$ is a derivation of $S$ and $z$ lies in the center of $R$, and (c) $g \otimes L_w$ where $g$ is a derivation of $R$ and $w$ lies in the center of $S$.


## 1. Introduction

Let $R$ and $S$ be nonassociative algebras. What are natural examples of derivations of the tensor product algebra $R \otimes S$? First of all, just as in any algebra, every element $u$ from the nucleus gives rise to the derivation $x \mapsto ux - xu$. Next, given a derivation $f$ of $S$ and an element $z$ from the center of $R$, the map given by $x \otimes y \mapsto zx \otimes f(y)$ is a derivation

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of $R \otimes S$. Similarly, $x \otimes y \mapsto g(x) \otimes wy$ defines a derivation of $R \otimes S$ for every derivation $g$ of $R$ and every central element $w \in S$. The goal of this short paper is to prove that under rather mild assumptions – namely, both $R$ and $S$ are unital and either one of them is finite dimensional or both are finitely generated – every derivation of $R \otimes S$ is the sum of derivations of the three types just described. From the nature of this result, and the relative simplicity of its proof, one would expect that it is known; however, we have not been able to find it in the literature. Among related results, we first mention the one by Block [3, Theorem 7.1] which considers a similar situation, just that the assumption that $R$ is unital is weakened and, on the other hand, $S$ is assumed to be associative and commutative. See also [1] for some extensions of Block's theorem. Benkart and Osborn dealt with the special case where $R$ is the (associative) matrix algebra $M_n(F)$ [2, Corollary 4.9]. Finally, in the case where both $R$ and $S$ are associative, the description of derivations of $R \otimes S$ can be (under some finiteness assumptions) obtained as a byproduct of results on Hochschild cohomology; see, for example, [7, Corollary 3.4].

In the next section we provide all definitions and prove a basic lemma. The third section is devoted to the main result, and in the last, fourth, section we record some corollaries.

## 2. Preliminaries

Let $A$ be a nonassociative (i.e., not necessarily associative) algebra over a field $F$. For $x, y, z \in A$ we write

$$[x, y, z] = (xy)z - x(yz).$$

The set

$$N(A) = \{n \in A \mid [n, A, A] = [A, n, A] = [A, A, n] = 0\}$$

is called the *nucleus* of $A$, and the set

$$Z(A) = \{z \in N(A) \mid zx = xz \text{ for all } x \in A\}$$

is called the *center* of $A$. Of course, $A$ is associative if and only if $N(A) = A$, and in this case the center is simply the set of elements that commute with all elements in $A$. We will consider the case where $A = R \otimes S$, the tensor product of unital algebras $R$ and $S$. It is therefore important to note that

$$N(R \otimes S) = N(R) \otimes N(S),$$

as one can readily check.

Recall that a linear map $d : A \to A$ is called a *derivation* if it satisfies