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Multiplicative perturbations of matrices and the generalized triple reverse order law for the Moore–Penrose inverse

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ABSTRACT

For a matrix A, let $A^\dagger$ denote its Moore–Penrose inverse. A matrix M is called a multiplicative perturbation of $T \in \mathbb{C}^{m\times n}$ if $M = ETF^*$ for some $E \in \mathbb{C}^{m\times m}$ and $F \in \mathbb{C}^{n\times n}$. Based on the alternative expression for M as $M = (ETT^\dagger)\cdot T\cdot(FT^\dagger T)^*$, the generalized triple reverse order law for the Moore–Penrose inverse is obtained as

$$M^\dagger = \left((FT^\dagger T)^*\right)^\dagger \cdot \left(YY^\dagger TZZ^\dagger\right)^\dagger_{LR^{-1}} \cdot (ETT^\dagger)^\dagger,$$

where $\left(YY^\dagger TZZ^\dagger\right)^\dagger_{LR^{-1}}$ is the weighted Moore–Penrose inverse for certain matrices Y, Z, L and R associated to the triple (T, E, F). Furthermore, it is proved that this weighted Moore–Penrose inverse in the resulting expression for $M^\dagger$ can be really replaced with $T^\dagger$ if

$$(ETT^\dagger)^\dagger ETT^\dagger \cdot T = T\cdot(FT^\dagger T)^\dagger(FT^\dagger T).$$

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In the special case that rank(M) = rank(T) or M is a weak perturbation of T, a simplified version of $M^\dagger$, as well as $MM^\dagger$ and $M^\dagger M$, is also derived.

1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{C}^{m\times n}$ denote the sets of positive integers and $m \times n$ complex matrices, respectively. The identity matrix and the zero matrix in $\mathbb{C}^{n\times n}$ are denoted by I_n and 0_n, respectively. For any $A \in \mathbb{C}^{m\times n}$, let A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the conjugate transpose, the range and the null space of A, respectively.

Definition 1.1. Let $A \in \mathbb{C}^{m\times n}, M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be given such that M and N are positive definite. The weighted Moore–Penrose inverse $A^\dagger_{MN}$ is the unique element $X \in \mathbb{C}^{n\times m}$ which satisfies

$$AXA = A, XAX = X, (MAX)^* = MAX \text{ and } (NXA)^* = NXA. \tag{1.1}$$

If $M = I_m$ and $N = I_n$, then $A^\dagger_{MN}$ is denoted simply by $A^\dagger$, which is called the Moore–Penrose inverse of A such that

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger \text{ and } (A^\dagger A)^* = A^\dagger A. \tag{1.2}$$

It is known [10] that

$$(A^\dagger)^* = (A^*)^\dagger,\ \mathcal{R}(A^\dagger) = \mathcal{R}(A^*) \text{ and } \mathcal{N}(A^\dagger) = \mathcal{N}(A^*). \tag{1.3}$$

If in addition $m = n$ and $A = A^*$, then $(A^\dagger)^* = A^\dagger$ and $AA^\dagger = A^\dagger A$.

The Moore–Penrose inverse has various applications in controllability problem, least squares problem, linear Glauber model, linear Hamiltonian system, stochastic signal and so on. One research field of the Moore–Penrose inverse is its representation theory associated to two kinds of perturbations, namely, the additive perturbation and the multiplicative perturbation. Much progress has been made in the case of the additive perturbation [6,11,13,14]. Yet, little has been done in the case of the multiplicative perturbation. The purpose of this paper is to set up the general theory of representations for the Moore–Penrose inverse in the latter case.

Let $T \in \mathbb{C}^{m\times n}$ be given. A matrix $M \in \mathbb{C}^{m\times n}$ is called a multiplicative perturbation of T if M can be expressed as

$$M = ETF^*, \text{ where } E \in \mathbb{C}^{m\times m} \text{ and } F \in \mathbb{C}^{n\times n}. \tag{1.4}$$