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Optimum skew energy of a tournament ☆

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ABSTRACT

Let G be a simple undirected graph and G^σ the corresponding oriented graph of G with the orientation σ. The skew energy of G^σ, denoted by $\varepsilon(G^\sigma)$, is defined as the sum of the singular values of its skew adjacency matrix $S(G^\sigma)$. In 2010, Adiga et al. proved $\varepsilon(G^\sigma) \le n\sqrt{\Delta}$, where Δ is the maximum degree of G of order n. In this paper, we characterize the skew energy of a tournament and present some properties about an optimum skew energy tournament.

1. Introduction

Let $G = (V(G), E(G))$ be a simple and undirected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. Denote by $\Delta = \Delta(G)$ the maximum degree of G. Let G^σ be an oriented graph of G, where σ is an orientation which assigns to each edge of G a direction. Then G is called the underlying graph of G^σ. A *tournament* K_n^σ is an oriented graph of a complete graph K_n of order n. In fact, K_n^σ is a $(n-1)$-regular oriented graph.

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Let $A(G)$ be the $(0,1)$-adjacency matrix of G. Let $S(G^\sigma) = [s_{ij}]$ be the skew-adjacency matrix of G^σ, where $s_{ij} = 1$ whenever $(v_i, v_j) \in E(G^\sigma)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in E(G^\sigma)$, $s_{ij} = 0$ otherwise. The skew energy $\varepsilon_s(G^\sigma)$ of G^σ is defined as the sum of the singular values of $S(G^\sigma)$. Actually $S(G^\sigma)$ is a skew-symmetric matrix, so the sum of the singular values of $S(G^\sigma)$ is the sum of the absolute values of its eigenvalues.

Gutman [7] introduced the energy of a simple undirected graph. Then there are a lot of results and achievements about the energy of the adjacent matrix of a graph. For details see [8] and [11].

Adiga et al. [1] raised the skew energy of an oriented graph firstly, and they obtained some results about skew energy of an oriented graph at the same time. In particular, a sharp upper bound of $\varepsilon_s(G^\sigma)$ was obtained, that is

$$\varepsilon_s(G^\sigma) \leq n\sqrt{\Delta},$$

where n is the order of G and Δ is the maximum degree of G. They proved $\varepsilon_s(G^\sigma) = n\sqrt{\Delta}$ if and only if $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$, where I_n is the identity matrix of order n. In the following, we call an oriented graph G^σ with n vertices and maximum degree Δ an *optimum skew energy oriented graph* if $\varepsilon_s(G^\sigma) = n\sqrt{\Delta}$. We call an *optimum skew energy tournament* especially for a tournament. At the same time the orientation σ is called the *optimum orientation.*

Which k-regular oriented graphs G^σ of order n satisfy $\varepsilon_s(G^\sigma) = n\sqrt{k}$?

Adiga et al. [1] solved the problem for situations of $k = 1$ and $k = 2$. Gong and Xu [5] characterized all 3-regular oriented graphs with optimum skew energy. Gong and Xu [6] considered the 4-regular oriented graphs with optimum skew energy. Chen et al. [2] concentrated on the 4-regular oriented graphs with optimum skew energy. The recent achievements about skew energy are collected in [10].

There is a connection between the above question and matrix theory. A weighing matrix $W = W(n; k)$ (see [3]) is defined as a square $(0, \pm 1)$ matrix with k non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore, we know $WW^T = kI_n$. In fact, the skew-adjacent matrix of an optimum skew energy oriented graph is a skew-symmetric weighing matrix which is related to Hadamard Matrix Conjecture, see [12].

Particularly, a conference matrix of order n is an $n \times n$ matrix $C = C(n; k)$ with diagonal entries 0 and off-diagonal entries ± 1 which satisfies $C^T C = (n-1)I_n$ (see [3]).

In this paper, we characterize an optimum skew energy tournament, which is related to the conference matrix. We attain some properties about an optimum skew energy tournament.

2. Some useful notions and lemmas

Throughout this paper, let G^σ be an oriented graph with n vertices and $\phi_s(G^\sigma, \lambda) = \sum_{i=0}^{n} c_i \lambda^{n-i}$ be the skew-characteristic polynomial of G^σ.