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On essentially semi regular linear relations

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A B S T R A C T

The characterization of bounded essentially semi regular operators in terms of Kato decomposition of finite type was studied by several authors. In this paper, we extend this characterization to the case of essentially semi regular linear relations. We also give other characterization of such linear relations. Further, we apply the obtained results to analyse the stability of the class of essentially semi regular linear relations under additional operator perturbations. Finally, as an application, we get some useful connections between the Fredholm spectrum and the essentially semi regular spectrum of linear relations.

1. Introduction

Bounded essentially semi regular operators in Banach spaces are well known and they were studied by many authors (see [1,12,18,22] and [23]). Later, the main results of the bounded case are extended to the case of closed operators (see [6] and [8]). The purpose

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of this paper is to consider the notion of essentially semi regular in the general setting of linear relations. It is shown that many of the results of Aiena [1], Müller [23] and other authors for operators remain valid in the context of linear relations.

Semi Fredholm operators on Banach spaces admit an important decomposition property introduced by Kato in 1958 [15]. This decomposition is known as the Kato decomposition. Then, this result was generalized by many authors and in many directions (see [14,19] and [21]). As an illustration, the classes of quasi Fredholm, regular, Kato type, semi Fredholm and B-Fredholm operators satisfy this decomposition. Also in [1], Aiena proved that the class of essentially semi regular operators is characterized by a Kato decomposition of finite type.

In this paper we extend this result to the case of multivalued linear operators in Banach spaces. For that, we introduce and investigate the notion of essentially semi regular linear relations and we proved that this new class is very useful for studying the Fredholm spectrum of linear relations.

Our paper is organized as follows:

In section 2, we give some results from the theory of linear relations which we will use in the next sections.

In section 3, we prove the important property of Kato decomposition of finite type of essentially semi regular linear relations. Then we give another characterization of such relations.

In section 4, we give a necessary and sufficient condition for which the polynomial of an essentially semi regular linear relation is an essentially semi regular linear relation.

Finally, the section 5 is devoted to the analysis of the stability of essentially semi regular linear relations under additional operator perturbations and as an application, we get some useful connections between the Fredholm spectrum and the essentially semi regular spectrum of linear relations.

2. Preliminary and auxiliary results

In this section we recall progressively some basic results of linear relations needed in the sequel, in the attempt of making our paper as self contained as possible. We followed the notation and terminology of the book [10].

We denote by X the linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation (or a multivalued linear operator) in X, $T : X \to X$ is a mapping from a subspace $D(T) \subseteq X$, called the domain of T, into the collection of nonempty subsets of X such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero α, β scalars and $x_1, x_2 \in D(T)$. For $x \in X \backslash D(T)$ we define $Tx = \emptyset$. With this convention, we have $D(T) = \{x \in X : \ Tx \neq \emptyset\}$. We denote the class of linear relations in X by $\mathcal{LR}(X)$. A linear relation in X is uniquely determined by its graph, $G(T)$, which is defined by $G(T) := \{(x, y) \in X \times X : x \in D(T), y \in Tx\}$, so that we can identify T with $G(T)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. The subspaces $T(0)$, $T^{-1}(0) := N(T)$ and $R(T) := T(D(T))$ are called the multivalued part, the null space and the range of T,