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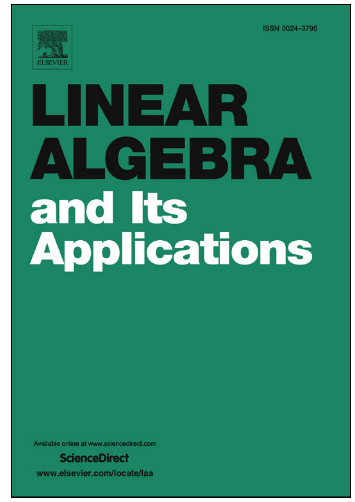

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GEOMETRIC MEAN FLOWS AND THE CARTAN BARYCENTER ON THE WASSERSTEIN SPACE OVER POSITIVE DEFINITE MATRICES

FUMIO HIAI AND YONGDO LIM

Dedicated to the memory of Professor Takayuki Furuta

Abstract. We introduce a class of flows on the Wasserstein space of probability measures with finite first moment on the Cartan-Hadamard Riemannian manifold of positive definite matrices, and consider the problem of differentiability of the corresponding Cartan barycentric trajectory. As a consequence we have a version of Lie-Trotter formula and a related unitarily invariant norm inequality. Furthermore, a fixed point theorem related to the Karcher equation and the Cartan barycentric trajectory is also presented as an application.

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1. Introduction and main theorem

Let $\mathbb{P}_m$ be the set of $m \times m$ positive definite matrices, which is a smooth Riemannian manifold with the *Riemannian trace metric* $\langle X, Y\rangle_A = \operatorname{tr} A^{-1}XA^{-1}Y$, where $A \in \mathbb{P}_m$ and $X, Y \in \mathbb{H}_m$, the Euclidean space of $m \times m$ Hermitian matrices equipped with the inner product $\langle X, Y\rangle = \operatorname{tr} XY$. Then $\mathbb{P}_m$ is a Cartan-Hadamard Riemannian manifold, a simply connected complete Riemannian manifold with non-positive sectional curvature (the canonical 2-tensor is non-negative). The Riemannian distance between $A, B \in \mathbb{P}_m$ with respect to the above metric is given by $d(A, B) = \|\log A^{-1/2}BA^{-1/2}\|_2$, where $\|X\|_2 = (\operatorname{tr} X^2)^{1/2}$ for $X \in \mathbb{H}_m$, and the unique (up to parametrization) geodesic joining A and B is given as the curve of *weighted geometric means*

$$t \in [0,1] \quad \longmapsto \quad A\#_t B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}. \tag{1.1}$$

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