Linear Algebra and its Applications 533 (2017) 210–234

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Convex and quasiconvex functions on trees and their applications

R.B. Bapat [a], D. Kalita [b], M. Nath [b,*], D. Sarma [b]

[a] *Stat-Math Unit, Indian Statistical Institute Delhi, 7-SJSS Marg, New Delhi 110 016, India*
[b] *Department of Mathematical Sciences, Tezpur University, Tezpur 784028, India*

ARTICLE INFO

Article history:
Received 6 December 2016
Accepted 10 July 2017
Available online 22 July 2017
Submitted by S. Fallat

MSC:
05C50
05C12

Keywords:
Convex functions
Distance matrix
Distance Laplacian
Distance signless Laplacian

ABSTRACT

We introduce convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. It is shown that the Perron vector of the distance matrix is strictly convex whereas the Perron vector of the distance signless Laplacian is quasiconvex for a tree. In the class of all trees with a given number of pendent vertices, we prove that the distance Laplacian and distance signless Laplacian spectral radius are both maximized at a dumbbell. Among all trees with fixed maximum degree, we prove that the broom is the unique tree that maximizes the distance Laplacian and distance signless Laplacian spectral radius. We find the unique graph that maximizes the distance spectral radius in the class of all unicyclic graphs of girth g on n vertices. Also we find the unique graph that maximizes the distance signless Laplacian and the distance Laplacian spectral radius in the class of all unicyclic graphs on n vertices.

* Corresponding author.
E-mail addresses: rbb@isd.ac.in (R.B. Bapat), kdebajit@tezu.ernet.in (D. Kalita), milan@tezu.ernet.in (M. Nath), deepaks@tezu.ernet.in (D. Sarma).

http://dx.doi.org/10.1016/j.laa.2017.07.012
0024-3795/

1. Introduction

All our graphs are finite, undirected, connected and simple. Let G be a graph on vertices $\{1, 2, \ldots, n\}$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. For $i, j \in V(G)$, the *distance* between i and j, denoted by $d_G(i, j)$ or simply d_{ij}, is the length of a shortest path from i to j in G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$ is the $n \times n$ matrix with (i, j)-th entry d_{ij}.

Like the Laplacian and signless Laplacian matrix of a graph, Aouchiche and Hansen[1] first introduced the distance Laplacian and distance signless Laplacian of a graph. The transmission, denoted by $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. Let $Tr(G)$ be the $n \times n$ diagonal matrix with i-th diagonal entry $Tr(i)$. The distance Laplacian and the distance signless Laplacian of G are defined as the matrix $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$, respectively, see [1].

For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we have

$$x^T \mathcal{D}^L(G)x = \sum_{1 \le i < j \le n} d_{ij}(x_i - x_j)^2 \text{ and } x^T \mathcal{D}^Q(G)x = \sum_{1 \le i < j \le n} d_{ij}(x_i + x_j)^2. \quad (1.1)$$

It follows that $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ are positive semidefinite. Thus the eigenvalues of distance Laplacian and distance signless Laplacian matrix of a graph are all real. The largest eigenvalue of the distance, distance Laplacian and the distance signless Laplacian matrix are known as the distance spectral radius, distance Laplacian spectral radius and distance signless Laplacian spectral radius, respectively. We use $\rho^{\mathcal{D}}$, ρ^L and $\rho^{\mathcal{Q}}$ to denote the distance, distance Laplacian and distance signless Laplacian spectral radius, respectively.

Note that for a connected graph G, both $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ are irreducible nonnegative matrices. Thus by the Perron–Frobenius Theorem, both $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are simple, and there is a positive eigenvector of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$, respectively. Such eigenvectors corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are called *Perron vector* of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$, respectively. By an eigenvector we mean a unit eigenvector.

Recently, the study of distance, distance Laplacian and distance signless Laplacian spectral radius has attracted many researchers. The problem of finding all graphs with maximal or minimal distance spectral radius among certain class of graphs has been studied extensively. For related results, one may refer to [2,4–6,9] and the references therein.

Subhi and Powers in [8] proved that for $n \ge 3$ the path P_n has the maximum distance spectral radius among trees on n vertices. Stevanović and Ilić in [7] generalized this result, and proved that among trees with fixed maximum degree Δ, the broom graph has maximum distance spectral radius and proved that the star S_n is the unique graph with minimal distance spectral radius among all trees on n vertices.

If vertices i and j are adjacent, we write $i \sim j$. Vertices i, j, k in graph G are said to be consecutive if j is adjacent to both i and k. An edge independent set of a graph G is a set of edges such that any two distinct edges of the set are not incident on a common vertex.