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Nonlinear oblique projections ☆

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A B S T R A C T

We construct nonlinear oblique projections along subalgebras of nilpotent Lie algebras in terms of the Baker–Campbell–Hausdorff multiplication. We prove that these nonlinear projections are real analytic on every Schubert cell of the Grassmann manifold whose points are the subalgebras of the nilpotent Lie algebra under consideration.

1. Introduction

The Grassmann manifold of linear subspaces of a finite-dimensional vector space plays an important role in linear algebra, operator theory, and differential geometry. In particular, the study of oblique projections and operator ranges can be transparently conducted

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from the perspective of that manifold and of its infinite-dimensional versions, as one can see for instance in [4], [3], [2], and [11].

The oblique projections are linear projection operators defined by decompositions of a vector space into a direct sum of two subspaces. In this paper we study a nonlinear version of the oblique projections, replacing the commutative vector addition of a vector space $\mathfrak{U}$ by a more general noncommutative group structure defined by a polynomial map $\mathfrak{U}\times\mathfrak{U}\to\mathfrak{U}$, $(X,Y)\mapsto X\cdot Y$ satisfying $(tX)\cdot(sX)=(t+s)X$ for all $t,s\in\mathbb{R}$ and $X\in\mathfrak{U}$. As we will recall below in Section 4, such a group structure turns $\mathfrak{U}$ into a nilpotent Lie group and coincides with the Baker–Campbell–Hausdorff multiplication defined by a uniquely determined Lie bracket on $\mathfrak{U}$. (This construction makes sense if $\mathfrak{U}$ is a Banach space and it then leads to some interesting problems, as discussed for instance in [6].) In this setting, the role of the Grassmann manifold is held by the set $\mathrm{Gr}^{\mathrm{alg}}(\mathfrak{U})$ of all subalgebras, rather than the linear subspaces of $\mathfrak{U}$. The natural nonlinear oblique projections along subalgebras defined in this way has been proved to be an important tool in representation theory of Lie groups (see [12] and [13]).

In the noncommutative framework outlined above, we study these generalized oblique projections along subalgebras of a nilpotent Lie algebra, and we establish their analyticity properties on suitable Schubert cells (Theorem 5.3). This is our main result here, and it was motivated by our recent research on the structure of C^*-algebras of nilpotent Lie groups. (See [7], [8] and [9].) We will briefly explain this motivation toward the end of the present paper, which is organized as follows: In Section 2 we discuss analyticity of linear oblique projections, using the Moore–Penrose inverse. Then, in Section 3 we establish some properties of the Schubert stratification of the Grassmann manifold, for later use. In Section 4 we briefly recall nilpotent Lie groups and algebras, and finally, in Section 5 we obtain our main result on analyticity of nonlinear oblique projections.

General notation For any finite-dimensional real vector space $\mathfrak{U}$ we denote by $\mathcal{B}(\mathfrak{U})$ its unital associative algebra of linear operators on $\mathfrak{U}$. If $\mathfrak{U}$ is endowed with a scalar product, and thus $\mathfrak{U}$ is a finite-dimensional real Hilbert space, we denote $\mathcal{P}(\mathfrak{U}):=\{P\in\mathcal{B}(\mathfrak{U})\mid P=P^2=P^*\}$, which is well known to be a compact real analytic submanifold of the real vector space $\mathcal{B}(\mathfrak{U})$. For every linear subspace $\mathcal{W}\subseteq\mathfrak{U}$ we denote by $P_{\mathfrak{W}}\in\mathcal{P}(\mathfrak{U})$ the orthogonal projection of $\mathfrak{U}$ onto $\mathfrak{W}$. The *Grassmann manifold* of $\mathfrak{U}$ is the set $\mathrm{Gr}(\mathfrak{U})$ of all linear subspaces of $\mathfrak{U}$. The map $\mathrm{Gr}(\mathfrak{U})\to\mathcal{P}(\mathfrak{U})$, $\mathfrak{W}\mapsto P_{\mathfrak{W}}$, is a bijection, and we endow $\mathrm{Gr}(\mathfrak{U})$ with the structure of a real analytic manifold that makes that bijection into a real analytic diffeomorphism.

For any integer $n\geq 1$ we denote by $\mathcal{P}_n$ the set of all subsets of $\{1,\dots,n\}$. For every $e\in\mathcal{P}_n$ we denote $\complement e:=\{1,\dots,n\}\setminus e$, and we write $e=\{j_1<\cdots<j_d\}$ if $e=\{j_1,\dots,j_d\}$ with $j_1<\cdots<j_d$.