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On the characterization of graphs by star complements ☆

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ABSTRACT

Let G be a graph of order n and μ be an adjacency eigenvalue of G with multiplicity $k \geq 1$. A star complement for μ in G is an induced subgraph of G of order $n-k$ with no eigenvalue μ. In this paper, all the regular graphs with $K_{1,1,t}$ as a star complement are determined. Also, the maximal graphs with $K_{1,1,t}$ $(t \neq 8, 9)$ as a star complement for the eigenvalue $\mu = 1$, and K_7 as a star complement for the eigenvalue $\mu = -2$ are described.

1. Introduction

We consider only simple graphs. Let G be a graph with vertex set $V(G) = \{1, 2, \dots, n\}$, and $A(G) = (a_{ij})$ be the adjacency matrix of G, where $a_{ij} = 1$ if vertex i is adjacent to vertex j, and 0 otherwise. When vertices i and j are adjacent, this will be indicated with $i \sim j$. The eigenvalues of G, denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, are just the eigenvalues

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of $A(G)$. For an eigenvalue μ, let $\mathcal{E}(\mu)$ denote the eigenspace $\{\boldsymbol{x} \in \mathbb{R} \mid A(G)\boldsymbol{x} = \mu\boldsymbol{x}\}$. For more details on graph spectra, see [3].

If $X \subseteq V(G)$, denote $V(G) \setminus X$ by $\overline{X}$. Let μ be an eigenvalue of G with multiplicity k. We say that X is a *star set* for μ in G if $|X| = k$ and μ is not an eigenvalue of $G - X$, where $G - X$ is the subgraph of G induced by $\overline{X}$. In this situation $H = G - X$ is called a *star complement* corresponding to μ (or a μ-basic subgraph of G in [9]). Some basic properties of star sets can be found in Chapter 7 of [4]. If $|H| = t$, then $|X| \leq \binom{t}{2}$ and this bound is best possible (see [2]).

The graphs with some special graphs as star complements have been discussed in the literature. The graphs (not necessarily regular) with the complete bipartite graph $K_{r,s}$ $(r+s > 2)$ as a star complement were discussed in [10]. Regular graphs with the complete tripartite graph $K_{r,r,r}$ as a star complement were discussed in [1]. And the regular graphs with the star $K_{1,s}$ as a star complement were discussed in [12]. In this paper, all the regular graphs with $K_{1,1,t}$ as a star complement are determined in Section 3.

It was proved in [7] that if $Y \subset X$, then induced subgraph $G - Y$ also has H as a star complement for μ. Moreover if any graph with H as a star complement for μ is an induced subgraph of such a graph G, then G is a maximal graph with star complement H for μ (we say that it is an H-maximal graph for μ). In general, there will be various different maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph. The maximal graphs with $K_{1,1,t}$ $(t \neq 8, 9)$ as a star complement for eigenvalue $\mu = 1$ are determined in Section 3. In [11], the maximal graphs with K_8 as a star complement have been characterized. Since K_7 and K_8 are the only two complete graphs which can be star complements for $\mu = -2$, then the maximal graphs with K_7 as a star complement for $\mu = -2$ will be discussed in Section 4.

2. Preliminaries

In this section we note properties of star sets and star complements that will be required in the sequel. The following result provides a characterization of star sets and star complements.

Theorem 2.1 *([4]). Let G be a graph with an eigenvalue μ of multiplicity k $(k > 0)$, and $X \subseteq V(G)$. Then the following statements are equivalent:*

(1) *X is a star set for μ;*
(2) *$\mathbb{R}^n = \mathcal{E}(\mu) \oplus \mathcal{V}$, where $\mathcal{V} = \langle \boldsymbol{e}_i : i \notin X \rangle$, $\boldsymbol{e}_i$ is the i-th column of identity matrix I;*
(3) *$\mathcal{E}(\mu)$ has basis $\{P\boldsymbol{e}_i : i \in X\}$, where P is the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$.*

Lemma 2.1 *([4]). Let X be a star set for μ in G and $\overline{X} = V(G) \setminus X$.*

(1) *If $\mu \neq 0$, then $\overline{X}$ is a dominating set for G;*
(2) *If $\mu \neq -1$ or 0, then $\overline{X}$ is a location-dominating set for G, that is, the $\overline{X}$-neighbourhoods of distinct vertices in X are distinct and non-empty.*