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# Equiangular line systems and switching classes containing regular graphs

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ABSTRACT

We develop the theory of equiangular lines in Euclidean spaces. Our focus is on the question of when a Seidel matrix having precisely three distinct eigenvalues has a regular graph in its switching class. We make some progress towards an answer to this question by finding some necessary conditions and some sufficient conditions. Furthermore, we show that the cardinality of an equiangular line system in 18 dimensional Euclidean space is at most 60.



## 1. Introduction and preliminaries

Let $d$ be a positive integer and let $\mathcal{L} = \{l_1, \ldots, l_n\}$ be a system of lines in $\mathbb{R}^d$, where each $l_i$ is spanned by a unit vector $\mathbf{v}_i$. The line system $\mathcal{L}$ is called **equiangular** if there exists a constant $\alpha > 0$ such that the inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \pm\alpha$ for all $i \neq j$. (The constant $\alpha$ is called the *common angle*.) Let $N(d)$ denote the maximum cardinality

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**Table 1**
Bounds for the sequence $N(d)$ for $2 \leqslant d \leqslant 23$.

| $d$ | 2 | 3 | 4 | 5 | 6 | 7–13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N(d)$ | 3 | 6 | 6 | 10 | 16 | 28 | 28–29 | 36 | 40–41 | 48–50 | 54–60 | 72–75 | 90–95 | 126 | 176 | 276 |

of a system of equiangular lines in dimension $d$. Determining values of the sequence $\{N(d)\}_{d\in\mathbb{N}}$ is a classical problem that has received much attention [11,16,12] and recently [3,5,8,9,13] there have been some improvements to the upper bounds for $N(d)$ for various values of $d$. One contribution of this article is to improve the upper bound for $N(18)$ showing that $N(18) \leqslant 60$. Furthermore, we show that certain Seidel matrices corresponding to systems of 60 equiangular lines in $\mathbb{R}^{18}$ each must contain in their switching classes a regular graph having four distinct eigenvalues (see Remark 5.12 below).

In Table 1, we give the currently known (including the improvement from this paper) values or lower and upper bounds for $N(d)$ for $d$ at most 23.

For tables of bounds for equiangular line systems in larger dimensions we refer the reader to Barg and Yu [3].

A **Seidel matrix** is a symmetric $\{0, \pm 1\}$-matrix $S$ with zero diagonal and all off-diagonal entries nonzero. It is well-known [16] that Seidel matrices and equiangular line systems (with positive common angle) are equivalent: for $i \neq j$, the inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = a$ (with $a \in \{-\alpha, \alpha\}$) if and only if the $(i, j)$th entry of its corresponding Seidel matrix is $a/\alpha$. Moreover, if $\mathcal{L}$ is an equiangular line system of $n$ lines in $\mathbb{R}^d$ with common angle $\alpha > 0$ then its corresponding Seidel matrix $S$ has smallest eigenvalue $-1/\alpha$ with multiplicity $n-d$. Conversely, given a Seidel matrix $S$ with smallest eigenvalue $-1/\alpha < 0$ whose multiplicity is $n - d$, we can construct an equiangular line system of $n$ lines in $\mathbb{R}^d$ with common angle $\alpha$.

Let $\mathcal{O}_n(\mathbb{Z})$ denote the orthogonal group generated by signed permutation matrices of order $n$ and let $X$ and $Y$ be two $\mathbb{Z}$-matrices of order $n$. We say that $X$ and $Y$ are **switching equivalent** (denoted $X \cong Y$) if $X = P^\top Y P$ for some matrix $P \in \mathcal{O}_n(\mathbb{Z})$. It is clear that two switching-equivalent matrices have the same spectrum. We will sometimes use the verb "to switch" to mean conjugation by diagonal matrices from $\mathcal{O}_n(\mathbb{Z})$.

The symbols $I$, $J$, and $O$ will (respectively) always denote the identity matrix, the all-ones matrix, and the all-zeros matrix; the order of each matrix should be clear from the context in which it is used, however, the order will sometimes be indicated by a subscript. We use $\mathbf{1}$ to denote the all-ones (column) vector.

Let $S$ be a Seidel matrix. Then $A = (J-I-S)/2$ is the adjacency matrix for a graph $\Gamma$, which we call the **underlying graph** of $S$. The set of underlying graphs of Seidel matrices that are switching equivalent to $S$ is called the **switching class** of $S$. Throughout this paper we use techniques from spectral graph theory and we refer to [4] for the necessary background.

We will be concerned with the question of when a Seidel matrix has a regular graph in its switching class. Using regular graphs to construct systems of equiangular lines is not new. Indeed, De Caen [7] used a family of previously studied regular graphs to give