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Quasi Kronecker products and a determinant formula

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We introduce an extension of the Kronecker product for matrices which retains many of the properties of the usual Kronecker product. As an application we study matrices over divisor-closed sets with multiplicative entries, and show how these are quasi Kronecker products over the primes of simpler matrices. In particular this gives a formula for the determinant of such matrices which combines and generalizes a number of previous results on Smith type determinants.

0. Introduction

In [5], inspired by a result of Codéca and Nair [2], it was shown that matrices with multiplicative entries indexed over a divisor set $D(k) = \{d \in \mathbb{N} : d|k\}$ factorize as Kronecker (or tensor) products; namely, for $f : \mathbb{N}^2 \to \mathbb{C}$ multiplicative (as a function of two variables) and $A = (f(m,n))_{m,n\in D(k)}$,

$$A = \bigotimes_{p|k} A_p$$

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where $A_p = (f(p^i, p^j))_{0\le i,j\le r}$ and $p^r||k$.[1] Since Kronecker products satisfy many useful properties, this makes is possible to deduce lots of information about A from the A_p like its eigenvalues, norm and determinant.

It is natural to enquire what we can say more generally about matrices $A_S = (f(m,n))_{m,n\in S}$ for some finite set $S \subset \mathbb{N}$, in particular when f is multiplicative. We find a natural condition on S is that it should be *divisor closed*; i.e. $n \in S$ implies $d \in S$ whenever $d|n$. For example $S = \{1, \ldots, N\}$, which gives the usual $N \times N$ truncation, is divisor closed. Determinants of matrices over divisor-closed sets have been discussed by many authors (see for example, [1], [3]), after the well-known Smith determinant from 1876 [6].

We show in this more general setting that A_S still factorizes over the primes in S as a type of pseudo-Kronecker product. This more general Kronecker product still retains a number of useful properties which we investigate here. In particular, we find linearity, commutativity and associativity are retained, even if multiplicativity fails. Furthermore, a neat formula for the determinant (already found in [4] for $S = \{1, \ldots, N\}$) and results on positive definiteness are obtained. As a consequence, we find a formula for $\det A_S$ whenever S is divisor-closed and f is multiplicative. This generalises a number of earlier results concerning determinants of arithmetical matrices over divisor-closed sets.

1. Quasi Kronecker products

Let $A = (a_{ij})$ be an $n\times n$ matrix, $B = (b_{ij})$ an $m\times m$ matrix, and $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ where $\max\{l_1, \ldots, l_n\} = m$. Let $B_{rs} = (b_{ij})_{i\le r, j\le s}$ denote the $r \times s$ truncation of B. If $r = s$ we simply write B_r. Put $L = l_1 + \cdots + l_n$. Define $A \otimes_l B$ to be the $L \times L$ matrix given by the block matrix

$$A \otimes_l B = (a_{ij} B_{l_i l_j})_{i,j\le n}. \tag{0.1}$$

Example 1. With $l = (3, 2)$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes_l \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c & 2a & 2b \\ d & e & f & 2d & 2e \\ g & h & i & 2g & 2h \\ 3a & 3b & 3c & 4a & 4b \\ 3d & 3e & 3f & 4d & 4e \end{pmatrix}.$$

Remarks 1.

(a) If the l_i are constant, say $l_i = m$ for all i, then $A \otimes_l B$ reduces to $A \otimes B$, the usual Kronecker product.

[1] Here $p^r||k$ means, as usual, that $p^r|k$ but $p^{r+1} \nmid k$.