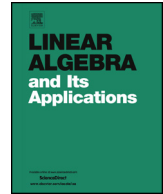




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# Linear Algebra and its Applications

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## Quasi Kronecker products and a determinant formula



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### ABSTRACT

We introduce an extension of the Kronecker product for matrices which retains many of the properties of the usual Kronecker product. As an application we study matrices over divisor-closed sets with multiplicative entries, and show how these are quasi Kronecker products over the primes of simpler matrices. In particular this gives a formula for the determinant of such matrices which combines and generalizes a number of previous results on Smith type determinants.

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## 0. Introduction

In [5], inspired by a result of Codéca and Nair [2], it was shown that matrices with multiplicative entries indexed over a divisor set  $D(k) = \{d \in \mathbb{N} : d|k\}$  factorize as Kronecker (or tensor) products; namely, for  $f : \mathbb{N}^2 \rightarrow \mathbb{C}$  multiplicative (as a function of two variables) and  $A = (f(m, n))_{m, n \in D(k)}$ ,

$$A = \bigotimes_{p|k} A_p$$

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where  $A_p = (f(p^i, p^j))_{0 \leq i, j \leq r}$  and  $p^r \nmid k$ .<sup>1</sup> Since Kronecker products satisfy many useful properties, this makes it possible to deduce lots of information about  $A$  from the  $A_p$  like its eigenvalues, norm and determinant.

It is natural to enquire what we can say more generally about matrices  $A_S = (f(m, n))_{m, n \in S}$  for some finite set  $S \subset \mathbb{N}$ , in particular when  $f$  is multiplicative. We find a natural condition on  $S$  is that it should be *divisor closed*; i.e.  $n \in S$  implies  $d \in S$  whenever  $d|n$ . For example  $S = \{1, \dots, N\}$ , which gives the usual  $N \times N$  truncation, is divisor closed. Determinants of matrices over divisor-closed sets have been discussed by many authors (see for example, [1], [3]), after the well-known Smith determinant from 1876 [6].

We show in this more general setting that  $A_S$  still factorizes over the primes in  $S$  as a type of pseudo-Kronecker product. This more general Kronecker product still retains a number of useful properties which we investigate here. In particular, we find linearity, commutativity and associativity are retained, even if multiplicativity fails. Furthermore, a neat formula for the determinant (already found in [4] for  $S = \{1, \dots, N\}$ ) and results on positive definiteness are obtained. As a consequence, we find a formula for  $\det A_S$  whenever  $S$  is divisor-closed and  $f$  is multiplicative. This generalises a number of earlier results concerning determinants of arithmetical matrices over divisor-closed sets.

## 1. Quasi Kronecker products

Let  $A = (a_{ij})$  be an  $n \times n$  matrix,  $B = (b_{ij})$  an  $m \times m$  matrix, and  $l = (l_1, \dots, l_n) \in \mathbb{N}^n$  where  $\max\{l_1, \dots, l_n\} = m$ . Let  $B_{rs} = (b_{ij})_{i \leq r, j \leq s}$  denote the  $r \times s$  truncation of  $B$ . If  $r = s$  we simply write  $B_r$ . Put  $L = l_1 + \dots + l_n$ . Define  $A \otimes_l B$  to be the  $L \times L$  matrix given by the block matrix

$$A \otimes_l B = (a_{ij} B_{l_i l_j})_{i, j \leq n}. \quad (0.1)$$

**Example 1.** With  $l = (3, 2)$ ,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes_l \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c & 2a & 2b \\ d & e & f & 2d & 2e \\ g & h & i & 2g & 2h \\ 3a & 3b & 3c & 4a & 4b \\ 3d & 3e & 3f & 4d & 4e \end{pmatrix}.$$

### Remarks 1.

- (a) If the  $l_i$  are constant, say  $l_i = m$  for all  $i$ , then  $A \otimes_l B$  reduces to  $A \otimes B$ , the usual Kronecker product.

<sup>1</sup> Here  $p^r \nmid k$  means, as usual, that  $p^r | k$  but  $p^{r+1} \nmid k$ .

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