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# Classification of three-dimensional evolution algebras

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We classify three dimensional evolution algebras over a field having characteristic different from 2 and in which there are roots of orders 2, 3 and 7.


## 1. Introduction

The use of non-associative algebras to formulate Mendel's laws was started by Etherington in his papers [6,7]. Other genetic algebras (those that model inheritance in

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genetics) called evolution algebras emerged to study non-Mendelian genetics. Its theory in the finite-dimensional case was introduced by Tian in [8]. The systematic study of evolution algebras of arbitrary dimension and of their algebraic properties was started in [1], where the authors analyze evolution subalgebras, ideals, non-degeneracy, simple evolution algebras and irreducible evolution algebras. The aim of this paper is to obtain the classification of three-dimensional evolution algebras having in mind to apply this classification in a near future in a biological setting and to detect possible tools to implement in wider classifications.

Two-dimensional evolution algebras over the complex numbers were determined in [3], although we have found that this classification is incomplete: the algebra $A$ with natural basis $\{e_1, e_2\}$ such that $e_1^2 = e_2$ and $e_2^2 = e_1$ is a two-dimensional evolution algebra not isomorphic to any of the six types in [3]. We realized of this fact when classifying the three-dimensional evolution algebras $A$ such that $\dim(A^2) = 2$ and having annihilator[1] of dimension 1.

The three dimensional case is much more complicated, as can be seen in this work, where we prove that there are 116 types of three-dimensional evolution algebras. The details can be found in [2, Tables 1–24]. Just after finishing this paper we found the article [5], where one of the aims of the authors is to classify indecomposable[2] nilpotent evolution algebras up to dimension five over algebraically closed fields of characteristic not two. The three-dimensional ones can be localized in our classification and for these, it is not necessary to consider algebraically closed fields.

In this paper we deal with evolution algebras over a field $\mathbb{K}$ of characteristic different from 2 and in which every polynomial of the form $x^n - k$, for $n = 2, 3, 7$ and $k \in \mathbb{K}$ has a root in the field. We denote by $\phi$ a seventh root of the unit and by $\zeta$ a third root of the unit.

In Section 2 we introduce the essential definitions. For every arbitrary finite dimensional algebra, fix a basis $B = \{e_i \mid i = 1, \dots, n\}$. The product of this algebra, relative to the basis $B$ is determined by the matrices of the multiplication operators, $M_B(\lambda_{e_i})$ (see (1)). The relationship under change of basis is also established. In the particular case of evolution algebras Theorem 2.2 shows this connection.

We start Section 3 by analyzing the action of the group $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathcal{M}_3(\mathbb{K})$. The orbits of this action will completely determine the non-isomorphic evolution algebras $A$ when $\dim(A^2) = 3$ and in some cases when $\dim(A^2) = 2$.

We have divided our study into four cases depending on the dimension of $A^2$, which can be 0, 1, 2 or 3. The first case is trivial. The study of the third and of the fourth ones is made by taking into account which are the possible matrices $P$ that appear as change of basis matrices. It happens that for dimension 3, as we have said, the only matrices are those in $S_3 \rtimes (\mathbb{K}^\times)^3$.

[1] The annihilator of $A$, $\mathrm{ann}(A)$, is defined as the set of those elements $x$ in $A$ such that $xA = 0$.
[2] Irreducible following [1].