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Inflation algorithm for loop-free non-negative edge-bipartite graphs of corank at least two

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We continue the study of finite connected loop-free edge-bipartite graphs Δ, with $m \geq 3$ vertices (a class of signed graphs), we started in Simson (2013) [48] and M. Gąsiorek et al. (2016) [19] by means of the non-symmetric Gram matrix $\check{G}_\Delta \in \mathbb{M}_m(\mathbb{Z})$ of Δ, its symmetric Gram matrix $G_\Delta := \frac{1}{2}[\check{G}_\Delta + \check{G}_\Delta^{tr}] \in \mathbb{M}_m(\frac{1}{2}\mathbb{Z})$, and the Gram quadratic form $q_\Delta : \mathbb{Z}^m \to \mathbb{Z}$. In particular, we study connected non-negative edge-bipartite graphs Δ, with $n + r \geq 3$ vertices of corank $r \geq 2$, in the sense that the symmetric Gram matrix $G_\Delta \in \mathbb{M}_{n+r}(\mathbb{Z})$ of Δ is positive semi-definite of rank $n \geq 1$. The edge-bipartite graphs Δ of corank $r \geq 2$ are studied, up to the weak Gram $\mathbb{Z}$-congruence $\Delta \sim_\mathbb{Z} \Delta'$, where $\Delta \sim_\mathbb{Z} \Delta'$ means that $G_{\Delta'} = B^{tr} \cdot G_\Delta \cdot B$, for some $B \in \mathbb{M}_{n+r}(\mathbb{Z})$ such that $\det B = \pm 1$. Our main result of the paper asserts that, given a connected edge-bipartite graph Δ with $n + r \geq 3$ vertices of corank $r \geq 2$, there exists a suitably chosen sequence $\mathbf{t}_*^-$ of the inflation operators of the form $\Delta' \mapsto \mathbf{t}_{ab}^-\Delta'$ such that the composite operator $\Delta \mapsto \mathbf{t}_*^-\Delta$ reduces Δ to a connected bigraph $\widehat{D}_n^{(r)}$ such that $\Delta \sim_\mathbb{Z} \widehat{D}_n^{(r)}$ and $\widehat{D}_n^{(r)}$ is one of the canonical r-vertex extensions $\widehat{\mathbb{A}}_n^{(r)}$, $n \geq 1$, $\widehat{\mathbb{D}}_n^{(r)}$, $n \geq 4$, $\widehat{\mathbb{E}}_6^{(r)}$, $\widehat{\mathbb{E}}_7^{(r)}$, and $\widehat{\mathbb{E}}_8^{(r)}$, with $n + r$ vertices, of the simply laced Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, with $n \geq 1$ vertices. The algorithm constructs also a matrix $B \in \mathbb{M}_{n+r}(\mathbb{Z})$, with

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$\det B = \pm 1$, defining the weak Gram $\mathbb{Z}$-congruence $\Delta \sim_{\mathbb{Z}} \widehat{D}_n^{(r)}$, that is, satisfying the equation $G_{\widehat{D}_n^{(r)}} = B^{tr} \cdot G_\Delta \cdot B$.

1. Introduction

We continue the study of loop-free edge-bipartite graphs Δ (i.e., a class of signed graphs with a separation property) we started in [48] and developed in [52] and [19]. Here we concentrate on a Gram classification of all loop-free edge-bipartite graphs that are non-negative of a fixed corank $r \geq 2$. A motivation for the study of edge-bipartite graphs is given in [48], where also a related algebraic framework is introduced.

Throughout, we freely use the terminology and notation introduced in [48,52,19]. We denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ the field of the rational, the real, and the complex numbers, respectively. Given a non-empty set I, we view $\mathbb{Z}^I$, as a free abelian group and, given $m \geq 1$, we view $\mathbb{Z}^m$ as a free abelian group. We denote by $e_1, \ldots, e_m$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^m$.

By $\mathbb{M}_m(\mathbb{Z})$ we denote the $\mathbb{Z}$-algebra of all square m by m matrices $A = [a_{ij}]$, with $a_{ij} \in \mathbb{Z}$, and by $E \in \mathbb{M}_m(\mathbb{Z})$ the identity matrix. The transpose of A is denoted by A^{tr}, and we set $A^{-tr} = (A^{tr})^{-1} = (A^{-1})^{tr}$, for $A \in \mathrm{Gl}(m, \mathbb{Z})$. The group

$$\mathrm{Gl}(m, \mathbb{Z}) := \{A \in \mathbb{M}_m(\mathbb{Z}),\ \det A \in \{-1, 1\}\} \subseteq \mathbb{M}_m(\mathbb{Z})$$

is called the (integral) **general linear group**. Given $A \in \mathbb{M}_m(\mathbb{Z})$ and $B \in \mathrm{Gl}(m, \mathbb{Z})$, we set $A * B := B^{tr} \cdot A \cdot B$. It is easy to check that the operation

$$* : \mathbb{M}_m(\mathbb{Z}) \times \mathrm{Gl}(m, \mathbb{Z}) \to \mathbb{M}_m(\mathbb{Z}), \quad (A, B) \mapsto A * B,$$

is a right group action of the group $\mathrm{Gl}(m, \mathbb{Z})$ on the abelian group $\mathbb{M}_m(\mathbb{Z})$.

By an **edge-bipartite graph** (for short: **bigraph**) with $m \geq 1$ vertices, we mean a signed (multi)graph (Δ, σ), where $\Delta = (\Delta_0, \Delta_1)$ is a (multi)graph with the finite set of vertices Δ_0 of cardinality $m \geq 1$ and the finite (multi)set of edges Δ_1, and $\sigma = (\sigma_0, \sigma_1)$ is a pair, with the sign map $\sigma_1 : \Delta_1 \to \{-1, +1\}$ and the labeling bijection $\sigma_0 : \Delta_0 \to \{1, 2, \ldots, m\}$, compare with [57]. Moreover, we assume that, given $a, b \in \Delta_0$, the sign map $\sigma_1 : \Delta_1(a, b) \to \{-1, +1\}$ is constant on the set $\Delta_1(a, b) = \Delta_1(b, a) \subset \Delta_1$ of all edges $\beta \in \Delta_1$ such that a and b are adjacent with β. In other words, $\sigma_1(\beta) = \sigma_1(\gamma)$, for all $\beta, \gamma \in \Delta_1(a, b)$. We set

$$\Delta_1^-(a, b) = \Delta_1^-(b, a) = \{\beta \in \Delta_1(a, b); \quad \sigma_1(\beta) = -1\},$$
$$\Delta_1^+(a, b) = \Delta_1^+(b, a) = \{\beta \in \Delta_1(a, b); \quad \sigma_1(\beta) = 1\}.$$