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Rational factorizations of completely positive matrices

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In this note it is proved that every rational matrix which lies in the interior of the cone of completely positive matrices also has a rational cp-factorization.

1. Introduction

The cone of completely positive matrices is central to copositive programming, see [3] and also for several topics in matrix theory, see [1]. However, so far, this cone is quite mysterious, many basic questions about it are open. In [2] Berman, Dür, and

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Shaked-Monderer ask: *Given a matrix $A \in \mathcal{CP}_n$ all of whose entries are integral, does A always have a rational cp-factorization?*

The *cone of completely positive matrices* is defined as the convex cone spanned by symmetric rank-1-matrices xx^T where x lies in the nonnegative orthant $\mathbb{R}^n_{\geq 0}$:

$$\mathcal{CP}_n = \operatorname{cone}\{xx^\mathsf{T} : x \in \mathbb{R}^n_{\geq 0}\}.$$

A *cp-factorization* of a matrix A is a factorization of the form

$$A = \sum_{i=1}^{m} \alpha_i x_i x_i^\mathsf{T} \quad \text{with } \alpha_i \geq 0 \text{ and } x_i \in \mathbb{R}^n_{\geq 0}, \quad \text{for } i = 1, \ldots, m.$$

We talk about a *rational cp-factorization* when the α_i's are rational numbers and when the x_i's are rational vectors. Of course, in a rational cp-factorization we can assume that the x_i's are integral vectors.

In this note we prove the following theorem:

Theorem 1.1. *Every rational matrix which lies in the interior of the cone of completely positive matrices has a rational cp-factorization.*

So to fully answer the question of Berman, Dür, and Shaked-Monderer, it remains to consider the boundary of $\mathcal{CP}_n$.

2. Proof of Theorem 1.1

For the proof we will need a classical result from simultaneous Diophantine approximation, a theorem of Dirichlet, which we state here. One can find a proof of Dirichlet's theorem for example in the book [4, Theorem 5.2.1] of Grötschel, Lovász, and Schrijver.

Theorem 2.1. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers and let ε be a real number with $0 < \varepsilon < 1$. Then there exist integers $p_1, \ldots, p_n$ and a natural number q with $1 \leq q \leq \varepsilon^{-n}$ such that*

$$\left|\alpha_i - \frac{p_i}{q}\right| \leq \frac{\varepsilon}{q} \quad \textit{for all } i = 1, \ldots, n.$$

The next lemma collects standard, easy-to-prove facts about convex cones. Let E be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let $K \subseteq E$ be a *proper convex cone*, which means that K is closed, has a nonempty interior, and satisfies $K \cap (-K) = \{0\}$. Its *dual cone* is defined as $K^* = \{y \in E : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$.

Lemma 2.2. *Let $K \subseteq E$ be a proper convex cone. Then,*

$$\operatorname{int}(K) = \{x \in E : \langle x, y \rangle > 0 \textit{ for all } y \in K^* \setminus \{0\}\}, \tag{1}$$