Linear Algebra and its Applications 523 (2017) 79–101

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Tropicalization of facets of polytopes ☆

Xavier Allamigeon [a,*], Ricardo D. Katz [b]

[a] INRIA and CMAP, École Polytechnique, CNRS, 91128 Palaiseau Cedex, France
[b] CONICET-CIFASIS, Bv. 27 de Febrero 210 bis, 2000 Rosario, Argentina

A R T I C L E   I N F O

Article history:
Received 4 July 2016
Accepted 5 February 2017
Submitted by P. Butkovič

MSC:
14T05
52A01

Keywords:
Tropical convexity
Polytopes
Facet-defining half-spaces
External representations
Puiseux series field
Hahn series field

A B S T R A C T

It is known that any tropical polytope is the image under the valuation map of ordinary polytopes over the Puiseux series field. The latter polytopes are called lifts of the tropical polytope. We prove that any pure tropical polytope is the intersection of the tropical half-spaces given by the images under the valuation map of the facet-defining half-spaces of a certain lift. We construct this lift explicitly, taking into account geometric properties of the given polytope. Moreover, when the generators of the tropical polytope are in general position, we prove that the above property is satisfied for any lift. This solves a conjecture of Develin and Yu.

☆ The authors were partially supported by the PGMO program of EDF and Fondation Mathématique Jacques Hadamard. An important part of this work was done during a visit of Ricardo D. Katz to the Ecole Polytechnique, which was funded by Digitéo. Ricardo D. Katz was also partially supported by CONICET Grant PIP 11220110101026.

* Corresponding author.
E-mail addresses: xavier.allamigeon@inria.fr (X. Allamigeon), katz@cifasis-conicet.gov.ar (R.D. Katz).

http://dx.doi.org/10.1016/j.laa.2017.02.011

1. Introduction

The *max-plus semiring* $\mathbb{R}_{\max}$, sometimes also referred to as the *tropical semiring*, consists of the set $\mathbb{R} \cup \{-\infty\}$ equipped with $x \oplus y := \max\{x, y\}$ as addition and $x \odot y := x + y$ as multiplication. The set $\mathbb{R}^n_{\max}$ is a semimodule over the max-plus semiring when equipped with the component-wise tropical addition $(u, v) \mapsto u \oplus v := (u_1 \oplus v_1, \dots, u_n \oplus v_n)$ and the tropical scalar multiplication $(\lambda, u) \mapsto \lambda \odot u := (\lambda \odot u_1, \dots, \lambda \odot u_n)$. For our purposes, it turns out to be more convenient to restrict ourselves to points with finite coordinates, and work in the *tropical projective space* $\mathbb{TP}^{n-1} := \mathbb{R}^n/(1, \dots, 1)\mathbb{R}$ (also known as *tropical projective torus* [17]), modding out by tropical scalar multiplication.

The tropical analogues of ordinary polytopes in the Euclidean space are defined as the tropical convex hulls of finite sets of points in the tropical projective space. More precisely, the *tropical polytope* generated by the points $v^1, \dots, v^p$ of $\mathbb{TP}^{n-1}$ is defined as

$$\mathcal{P} = \mathsf{tconv}\left(\{v^1, \dots, v^p\}\right) := \left\{(\lambda_1 \odot v^1) \oplus \cdots \oplus (\lambda_p \odot v^p) \in \mathbb{TP}^{n-1} \mid \lambda_1, \dots, \lambda_p \in \mathbb{R}\right\} . \tag{1}$$

A tropical polytope is said to be *pure* when it coincides with the closure of its interior (with respect to the topology of $\mathbb{TP}^{n-1}$ induced by the usual topology of $\mathbb{R}^n$). We refer to Fig. 1 below for an example (for visualization purposes, we represent the point $(x_1, \dots, x_n)$ of $\mathbb{TP}^{n-1}$ by the point $(x_2 - x_1, \dots, x_n - x_1)$ of $\mathbb{R}^{n-1}$).

The interest in tropical polytopes (also known as semimodules or max-plus cones), and more generally in tropically convex sets, comes from different areas, which include optimization [19], idempotent functional analysis [15] and control theory [9]. They are also interesting combinatorial objects, with connections to, for example, subdivisions of a

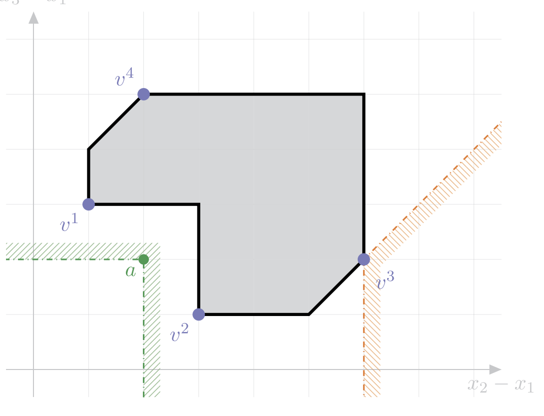

Fig. 1. A pure tropical polytope (in grey), and the half-space $\mathcal{H}(a, I)$ with $a = (0, 2, 2)$ and $I = \{2, 3\}$ (in green). The extreme points $v^1, \dots, v^4$ of the polytope are depicted in blue. The point $v^3 = (0, 6, 2)$ is extreme of type 2 because it is the only point of the polytope contained in the sector $\mathscr{S}(v^3, 2)$ (in orange). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)