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An improved upper bound for the number of distinct eigenvalues of a matrix after perturbation

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ABSTRACT

An upper bound for the number of distinct eigenvalues of a perturbed matrix has been recently established by P. E. Farrell [1, Theorem 1.3]. The estimate is the central result in Farrell's work and can be applied to estimate the number of Krylov iterations required for solving a perturbed linear system. In this paper, we present an improved upper bound for the number of distinct eigenvalues of a matrix after perturbation. Furthermore, some results based on the improved estimate are presented.

1. Introduction

The spectrum of a matrix after perturbation has been investigated by many authors. However, most work is devoted to discussing some special cases, especially the case of symmetric rank-one perturbations; see, for instance, [2–5]. Recently, P. E. Farrell [1] presented an upper bound for the number of distinct eigenvalues of arbitrary matrices perturbed by updates of arbitrary rank. Let $\mathbb{C}^{n\times n}$, $\Lambda(\cdot)$, $\operatorname{rank}(\cdot)$, and $|\cdot|$ be the set of

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all $n \times n$ complex matrices, the set of all distinct eigenvalues of a matrix, the rank of a matrix, and the cardinality of a set, respectively. Let $A, B \in \mathbb{C}^{n\times n}$, and let $C = A + B$. It is proved by Farrell [1, Theorem 1.3] that

$$|\Lambda(C)| \le (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A), \tag{1.1}$$

where $d(\cdot)$ denotes the defectivity of a matrix (see Definition 2.2 below). The result can be used to estimate the number of Krylov iterations for solving a linear system.

It follows from the definitions of $\Lambda(\cdot)$ and $|\cdot|$ that $|\Lambda(M)| \le n$ for all $M \in \mathbb{C}^{n\times n}$. Given $A \in \mathbb{C}^{n\times n}$, we can observe that the estimate (1.1) is mainly of interest in the situation that $\operatorname{rank}(B)$ is small, that is to say, B is a low-rank perturbation. More specifically, if $\operatorname{rank}(B) \le \frac{n-d(A)}{|\Lambda(A)|} - 1$, then $(\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A)$ $(\le n)$ is an applicable upper bound. On the other hand, if $\operatorname{rank}(B) > \frac{n-d(A)}{|\Lambda(A)|} - 1$, then $(\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A)$ $(> n)$ is a trivial upper bound.

Nevertheless, the estimate (1.1) is not sharp in certain cases. We now give a specific example to illustrate the defect of (1.1). Let λ_0 be an arbitrary complex number. We choose a matrix A as follows:

$$A = \begin{pmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0 & 1 \\ 0 & 0 & \cdots & 0 & \lambda_0 \end{pmatrix},$$

which is an $n \times n$ Jordan block. Thus, $|\Lambda(A)| = 1$ and $d(A) = n - 1$. Let $n \times n$ matrix B_r with rank r $(1 \le r \le n - 1)$ be defined by

$$B_r = \begin{pmatrix} T_r & O_{r\times(n-r-1)} \\ O_{(n-r)\times(r+1)} & O_{(n-r)\times(n-r-1)} \end{pmatrix}, \quad T_r = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & r & -1 \end{pmatrix}.$$

Hence, the upper bound in (1.1) is $(\operatorname{rank}(B_r) + 1)|\Lambda(A)| + d(A) = n + r > n$. In this case, the upper bound in (1.1) is always invalid (i.e., the upper bound is strictly greater than order n) for all $1 \le r \le n - 1$.

In this paper, we give an improved upper bound for the number of distinct eigenvalues of a matrix after perturbation. Under the same assumptions, we establish that

$$|\Lambda(C)| \le (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C). \tag{1.2}$$

Applying (1.2) to the above example, we can derive that the improved upper bound of $|\Lambda(C_r)|$ (here $C_r = A+B_r$) is $(\operatorname{rank}(B_r)+1)|\Lambda(A)|+d(A)-d(C_r) = (n+r)-(n-1-r) = 2r + 1$, which is an applicable upper bound, especially in low-rank perturbations.