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On positivity of spectral shift functions

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ABSTRACT

We show that every spectral shift function of an even order η_{2k} is nonnegative outside the convex hull of the spectrum of an initial operator $\operatorname{cvh}\sigma(H)$; every spectral shift function of an odd order η_{2k-1} is nonnegative (respectively, nonpositive) outside $\operatorname{cvh}\sigma(H)$ whenever a perturbation is nonnegative (respectively, nonpositive). We also derive several sufficient conditions for positivity of η_{2k} and η_{2k-1} on the whole real line.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space; let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{S}^n$ the nth Schatten class of operators in $\mathcal{B}(\mathcal{H})$. Let $\sigma(H)$ denote the spectrum of $H \in \mathcal{B}(\mathcal{H})$ and $\operatorname{cvh}\sigma(H)$ the convex hull of the set $\sigma(H)$. The following result is established in [4,3], and [5] for $n = 1$, $n = 2$, and $n \geq 3$, respectively.

Theorem 1.1. *Let $H = H^* \in \mathcal{B}(\mathcal{H})$ and $V = V^* \in \mathcal{S}^n$. Then, there exists a unique $\eta_n \in L^1(\mathbb{R})$, called the spectral shift function of order n, such that*

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$$\mathrm{Tr}\left(f(H+V)-f(H)-\sum_{k=0}^{n-1}\frac{1}{k!}\frac{d^k}{dt^k}\Big|_{t=0}f(H+tV)\right)=\int_{\mathbb{R}}f^{(n)}(t)\,\eta_n(t)\,dt, \quad (1.1)$$

for every $f \in C^{n+1}(\mathbb{R})$.

It is well known that η_2 is positive (≥ 0) and that η_1 for a sign-definite perturbation V has the same sign as V does. These facts can be derived from [1]; they are also proved via a multiple operator integration approach in Proposition 3.3. Sign-definiteness of η_1 is also discussed in [7, Section 8.2, Theorem 1]. In this paper, we address the following questions on sign-definiteness of higher order spectral shift functions.

Question 1.2. *Let $H = H^* \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$ be even, and $V = V^* \in \mathcal{S}^n$. Is η_n positive?*

Question 1.3. *Let $H = H^* \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$ be odd, and $V = V^* \in \mathcal{S}^n$ be positive (respectively, negative). Is η_n positive (respectively, negative)?*

It is simple to see via the multiple operator integration approach that Questions 1.2 and 1.3 have affirmative answers for every $n \in \mathbb{N}$ when H and V commute (see Proposition 3.4). We establish the following partial answers to these questions in the case of noncommuting H and V.

Theorem 1.4. *Let $H = H^* \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, and $V = V^* \in \mathcal{S}^n$. Denote $[a, b] = \mathrm{cvh}\,\sigma(H)$. Then, the following assertions hold.*

(i) If n is even, then $\eta_n(t) \geq 0$ for a.e. $t \in \mathbb{R}\backslash[a, b]$.
(ii) If n is odd and $V \geq 0$ (respectively, $V \leq 0$), then $\eta_n(t) \geq 0$ (respectively, $\eta_n(t) \leq 0$) for a.e. $t \in \mathbb{R}\backslash[a, b]$.
(iii) Let $n \in \mathbb{N}$. If $V \geq 0$, then $\eta_n(t) = 0$ for a.e. $t < a$; if $V \leq 0$, then $\eta_n(t) = 0$ for a.e. $t > b$.

Let $H = H^* \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, $V = V^* \in \mathcal{S}^n$, and $t \in [0, 1]$ be fixed and let E_t denote the spectral measure of $H + tV$. Let $\omega_t^{(n)}$ denote the set function

$$\omega_t^{(n)}(A_1, A_2, \dots, A_n) := \mathrm{Tr}\left(E_t(A_1)VE_t(A_2)V \dots E_t(A_n)V\right), \quad (1.2)$$

where $A_1, A_2, \dots, A_n$ are Borel subsets of $\mathbb{R}$. When $V \in \mathcal{S}^2$, the set function $\omega_t^{(n)}$ extends to a finite Borel measure on $\mathbb{R}^n$ (see, e.g., [2, Theorem 4.1 and Remark 4.2]).

The following theorem gives a sufficient condition for positivity of η_n on $\mathbb{R}$.

Theorem 1.5. *Let $H = H^* \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, and $V = V^* \in \mathcal{S}^n$. If for a.e. $t \in [0, 1]$, $\omega_t^{(n)}(A_1, \dots, A_n) \geq 0$ for all Borel subsets $A_1, \dots, A_n$ of $\mathbb{R}$, then $\eta_n \geq 0$.*