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Dilations, wandering subspaces, and inner functions

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The objective of this paper is to study wandering subspaces for commuting tuples of bounded operators on Hilbert spaces. It is shown that, for a large class of analytic functional Hilbert spaces $\mathcal{H}_K$ on the unit ball in $\mathbb{C}^n$, wandering subspaces for restrictions of the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_n})$ can be described in terms of suitable $\mathcal{H}_K$-inner functions. We prove that $\mathcal{H}_K$-inner functions are contractive multipliers and deduce a result on the multiplier norm of quasi-homogeneous polynomials as an application. Along the way we prove a refinement of a result of Arveson on the uniqueness of minimal dilations of pure row contractions.

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1. Introduction

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators on a complex Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{W} \subset \mathcal{H}$ is called a *wandering subspace* for T if

$$\mathcal{W} \perp T^{\boldsymbol{k}}\mathcal{W} \qquad (\boldsymbol{k} \in \mathbb{N}^n \setminus \{\boldsymbol{0}\}).$$

We say that $\mathcal{W}$ is a *generating wandering subspace* for T if in addition

$$\mathcal{H} = \overline{\text{span}}\{T^{\boldsymbol{k}}\mathcal{W} : \boldsymbol{k} \in \mathbb{N}^n\}.$$

Wandering subspaces were defined by Halmos in [10]. One of the main observations from [10] is the following. Let $\mathcal{E}$ be a Hilbert space and let $M_z : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ be the operator of multiplication with the argument on the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ on the unit disc $\mathbb{D}$. Suppose that $\mathcal{S}$ is a non-trivial closed M_z-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Then

$$\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$$

is a wandering subspace for $M_z|_{\mathcal{S}}$ such that

$$M_z^p\mathcal{W} \perp M_z^q\mathcal{W}$$

for all $p \neq q$ in $\mathbb{N}$ and

$$\mathcal{S} = \overline{\text{span}}\{z^m\mathcal{W} : m \in \mathbb{N}\}.$$

Hence

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} z^m\mathcal{W}$$

and up to unitary equivalence

$$M_z|_{\mathcal{S}} \text{ on } \mathcal{S} \cong M_z \text{ on } H^2_{\mathcal{W}}(\mathbb{D}).$$

In particular, we have $\mathcal{S} = V(H^2_{\mathcal{W}}(\mathbb{D}))$, where $V : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ is an isometry and $VM_z = M_zV$. One can show (see Lemma V.3.2 in [14] for details and more precise references) that any such intertwining isometry V acts as the multiplication operator $V = M_\Theta : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$, $f \mapsto \Theta f$, with a bounded analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E})}(\mathbb{D})$ such that Θ possesses isometric boundary values almost everywhere. In this case