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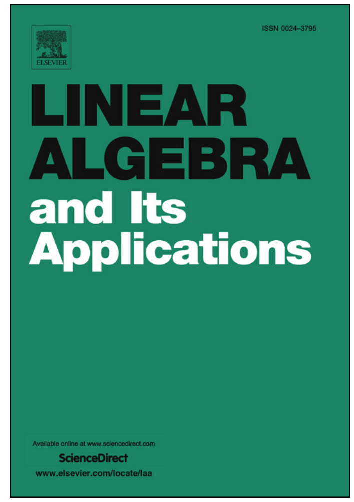

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A matricial view of the Karpelevič Theorem

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Abstract

The question of the exact region in the complex plane of the possible single eigenvalues of all n-by-n stochastic matrices was raised by Kolmogorov in 1937 and settled by Karpelevič in 1951 after a partial result by Dmitriev and Dynkin in 1946. The Karpelevič result is unwieldy, but a simplification was given by Đoković in 1990 and Ito in 1997. The Karpelevič region is determined by a set of boundary arcs each connecting consecutive roots of unity of order less than n. It is shown here that each of these arcs is realized by a single, somewhat simple, parameterized stochastic matrix. Other observations are made about the nature of the arcs and several further questions are raised. The doubly stochastic analog of the Karpelevič region remains open, but a conjecture about it is amplified.

Keywords: Stochastic matrix, Doubly stochastic matrix, Karpelevič arc, Karpelevič region, Ito polynomial, Realizing matrix
2010 MSC: 15A18, 15A29, 15B51

1. Introduction

In [11], Kolmogorov posed the problem of characterizing the subset of the complex plane, denoted by Θ_n, that consists of the individual eigenvalues of all n-by-n stochastic matrices.

One can easily verify that for each $n \geq 2$, the region Θ_n is closed, inscribed in the unit-disc, star-convex (with star-centers at zero and one), and symmetric with respect to the real-axis. Furthermore, it is clear that $\Theta_n \subseteq \Theta_{n+1}$, $\forall n \in \mathbb{N}$. In view of these properties, $\partial\Theta_n = \{\lambda \in \Theta_n : \alpha\lambda \notin \Theta_n, \forall \alpha > 1\}$, and each region is determined by its boundary.

Dmitriev and Dynkin [2] obtained a partial solution to Kolmogorov's problem, and Karpelevič [10, Theorem B], expanding on the work of [2], resolved

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