Linear Algebra and its Applications 520 (2017) 44–66

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Operator norm inequalities between tensor unfoldings on the partition lattice

Miaoyan Wang [a], Khanh Dao Duc [a], Jonathan Fischer [b], Yun S. Song [a,b,c,*]

[a] Department of Mathematics, University of Pennsylvania, United States
[b] Department of Statistics, University of California, Berkeley, United States
[c] Computer Science Division, University of California, Berkeley, United States

ARTICLE INFO

Article history:
Received 17 March 2016
Accepted 14 January 2017
Available online 17 January 2017
Submitted by L.-H. Lim

MSC:
15A60
15A69
06B99
05A18

Keywords:
Higher-order tensors
General unfoldings
Partition lattice
Operator norm
Orthogonality

ABSTRACT

Interest in higher-order tensors has recently surged in data-intensive fields, with a wide range of applications including image processing, blind source separation, community detection, and feature extraction. A common paradigm in tensor-related algorithms advocates unfolding (or flattening) the tensor into a matrix and applying classical methods developed for matrices. Despite the popularity of such techniques, how the functional properties of a tensor changes upon unfolding is currently not well understood. In contrast to the body of existing work which has focused almost exclusively on matricizations, we here consider all possible unfoldings of an order-k tensor, which are in one-to-one correspondence with the set of partitions of $\{1,\dots,k\}$. We derive general inequalities between the l^p-norms of arbitrary unfoldings defined on the partition lattice. In particular, we demonstrate how the spectral norm ($p=2$) of a tensor is bounded by that of its unfoldings, and obtain an improved upper bound on the ratio of the Frobenius norm to the spectral norm of an arbitrary tensor. For specially-structured tensors satisfying a generalized definition of orthogonal decomposability, we prove that

* Corresponding author.
E-mail address: yss@berkeley.edu (Y.S. Song).

http://dx.doi.org/10.1016/j.laa.2017.01.017

the spectral norm remains invariant under specific subsets of unfolding operations.

1. Introduction

Tensors of order 3 or greater, known as higher-order tensors, have recently attracted increased attention in many fields across science and engineering. Methods built on tensors provide powerful tools to capture complex structures in data that lower-order methods may fail to exploit. Among numerous examples, tensors have been used to detect patterns in time-course data [7,17,22,29] and to model higher-order cumulants [1,2,14]. However, tensor-based methods are fraught with challenges. Tensors are not simply matrices with more indices; rather, they are mathematical objects possessing multilinear algebraic properties. Indeed, extending familiar matrix concepts such as norms to tensors is non-trivial [12,18], and computing these quantities has proven to be NP-hard [4–6].

The spectral relations between a general tensor and its lower-order counterparts have yet to be studied. There are generally two types of approaches underlying many existing tensor-based methods. The first approach flattens the tensor into a matrix and applies matrix-based techniques in downstream analyses, notably higher-order SVD [3,10] and TensorFace [25]. Flattening is computationally convenient because of the ubiquity of well-established matrix-based methods, as well as the connection between tensor contraction and block matrix multiplication [19]. However, matricization leads to a potential loss of the structure found in the original tensor. This motivates the key question of how much information a flattening retains from its parent tensor.

The second approach either handles the tensor directly or unfolds it into objects of order-3 or higher. Recent work on bounding the spectral norm of sub-Gaussian tensors reveals that solving for the convex relaxation of tensor rank by unfolding is suboptimal [24]. Interestingly, in the context of tensor completion, unfolding a higher-order tensor into a nearly cubic tensor requires smaller sample sizes than matricization [28]. These results are probabilistic in nature and merely focus on a particular class of tensors. Assessing the general impact of unfolding operations on an arbitrary tensor and the role of the tensor's intrinsic structure remains challenging.

The primary goal of this paper is to study the effect of unfolding operations on functional properties of tensors, where an unfolding is any lower-order representation of a tensor. We study the operator norm of a tensor viewed as a multilinear functional because this quantity is commonly used in both theory and applications, especially in tensor completion [15,28] and low-rank approximation problems [23,27]. Given an order-k tensor, we represent each possible unfolding operation using a partition π of $[k] = \{1, \dots, k\}$, where a block in π corresponds to the set of modes that should be combined into a single mode. Each unfolding is a rearrangement of the elements of the original tensor