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Note on the free sets and free subsemimodules in semimodules ☆

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Some conditions that a finite set in an $\mathcal{L}$-semimodule is free are established. This gives an answer to an open problem raised by Tan (2016) in his work [7].

1. Introduction

The study of semimodules over semirings has a long history. In 1966, Yusuf [9] introduced the concept of an inverse semimodule over a semiring and obtained some analogues to theorems in module theory for inverse semimodules (note that an inverse semimodule M is a semimodule in which the monoid $(M, +)$ is an inverse semigroup). Since then,

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a number of works on semimodule theory were published (see e.g. [1–3,7]). We know that a semimodule structure is the one which arises naturally in the properties of sets of vectors with entries in a semiring. Thus, they turn out to be analogues for algebraic structure on semirings to the concept of a module for rings. However, we must be careful since not all properties are transferred in a straightforward way. For example, in general, a system of linearly independent vectors cannot be extended to a basis of a semimodule (see [2], Theorem 4). Some of facts known about free sets in modules have not yet been proved in semimodule, one of them is for a given finitely generated free semimodule over a commutative semiring, find a necessary and sufficient condition for a nonempty finite subset in this semimodule to be free (see [7], Open problem 3.1).

In this note, we give conditions under which a nonempty finite subset in a free semimodule is free. So that we give an answer to Open problem 3.1 in [7].

2. Definition and previous results

In this section, we collect only some necessary notions for the presentations of the main result in the next section.

Definition 2.1 *(Golan [3]).* A semiring $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ is an algebraic structure with the following properties:

(i) $(L, +, 0)$ is a commutative monoid,
(ii) $(L, \cdot, 1)$ is a monoid,
(iii) $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(s + t) \cdot r = s \cdot r + t \cdot r$ hold for all $r, s, t \in L$,
(iv) $0 \cdot r = r \cdot 0 = 0$ holds for all $r \in L$,
(v) $0 \neq 1$.

A semiring $\mathcal{L}$ is commutative if $r \cdot r' = r' \cdot r$ for all $r, r' \in L$.

Definition 2.2 *(Golan [3]).* Let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be a semiring. A left $\mathcal{L}$-semimodule is a commutative monoid $(\mathcal{M}, +)$ with additive identity $\mathbf{0}$ for which we have a function $L \times \mathcal{M} \to \mathcal{M}$, denoted by $(\lambda, \mathbf{a}) \mapsto \lambda\mathbf{a}$ and called a scalar multiplication, which satisfies the following conditions for all λ, μ in L and $\mathbf{a}, \mathbf{b}$ in $\mathcal{M}$:

(i) $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$,
(ii) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$,
(iii) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$,
(iv) $1\mathbf{a} = \mathbf{a}$,
(v) $\lambda\mathbf{0} = \mathbf{0} = 0\mathbf{a}$.

The definition of a right $\mathcal{L}$-semimodule is analogous. In this paper, $\mathcal{L}$-semimodules will always mean left $\mathcal{L}$-semimodules. $\mathcal{L}$-semimodules were studies in [2,5] under the name $\mathcal{L}$-semilinear spaces.