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Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples

Shaun Fallat [a,*], Charles R. Johnson [b], Alan D. Sokal [c,1]

[a] *Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada*
[b] *Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA*
[c] *Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA*

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We show that, for Hankel matrices, total nonnegativity (resp. total positivity) of order r is preserved by sum, Hadamard product, and Hadamard power with real exponent $t \geq r - 2$. We give examples to show that our results are sharp relative to matrix size and structure (general, symmetric or Hankel). Some of these examples also resolve the Hadamard critical-exponent problem for totally positive and totally nonnegative matrices.

* Corresponding author.
E-mail addresses: Shaun.Fallat@uregina.ca (S. Fallat), crjohn@wm.edu (C.R. Johnson), sokal@nyu.edu (A.D. Sokal).

[1] Also at Department of Mathematics, University College London, London WC1E 6BT, United Kingdom.

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1. Introduction

A matrix M of real numbers is called *totally nonnegative* (TN) if every minor of M is nonnegative, and *totally positive* (TP) if every minor of M is positive. More generally, M is called *totally nonnegative of order* r (TN_r) if every minor of M of size $\le r$ is nonnegative, and *totally positive of order* r (TP_r) if every minor of M of size $\le r$ is positive.[2] Of course, for m-by-n matrices, $\mathrm{TN} = \mathrm{TN}_r$ and $\mathrm{TP} = \mathrm{TP}_r$ where $r = \min(m, n)$. Background information on totally nonnegative and totally positive matrices and their applications can be found in [2,9,14,16,24,27].

It is an immediate consequence of the Cauchy–Binet formula that the product of two TN_r (resp. TP_r) matrices is TN_r (resp. TP_r). However, other natural matrix operations do not in general preserve total nonnegativity. For instance, it is well known (and easy to see by example) that the sum of two TP matrices need not even be TN_2. The situation is slightly (but not much) better when the matrices are symmetric. Likewise, it has been known for over 40 years that the Hadamard (entrywise) product of two TN (resp. TP) matrices is always TN_2 (resp. TP_2) but need not be TN_3 [26, p. 163]. Once again, the situation is slightly (but not much) better when the matrices are symmetric. In this paper we shall give counterexamples illustrating the various possibilities and showing the sharpness of each positive result.

The situation changes radically, however, for Hankel matrices, i.e. square matrices $A = (a_{ij})$ in which a_{ij} depends only on $i + j$. The Hankel matrices form an important subclass of symmetric matrices, and they arise in numerous applications [13,18,21,29, 31,32]. It is easy to see (Lemma 2.7 below) that a matrix is Hankel if and only if every contiguous submatrix is symmetric. Here we will exploit this fact to show that, for Hankel matrices, total nonnegativity — and more generally, total nonnegativity of order r — *is* preserved by sum and by Hadamard product. We will also show that total nonnegativity of order r is preserved under Hadamard powers with an arbitrary real exponent $t \ge r-2$.

One important motivation for this investigation was the connection between the Stieltjes moment problem [1,31] and the total positivity of Hankel matrices. It is well known that an *infinite* Hankel matrix $A = (a_{i+j})_{i,j=0}^{\infty}$ is totally nonnegative if and only if the underlying sequence $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ is a Stieltjes moment sequence (i.e. the moments of a positive measure on $[0, \infty)$): “only if” follows immediately from the standard positive-definiteness criterion for Stieltjes moment sequences [31, Theorem 1.3], while “if” follows by a simple Vandermonde-matrix argument [15, p. 460, Théorème 9], [27, Theorem 4.4]. This equivalence immediately implies that, for *infinite* Hankel matrices, total nonnegativity is preserved by sum and by Hadamard product. We therefore wondered whether the same result would hold when infinite Hankel matrices are replaced by finite ones, or

[2] **Warning:** Some authors (e.g. [2,24,27,30,32]) use the terms “totally positive” and “strictly totally positive” for what we have termed “totally nonnegative” and “totally positive”, respectively. So it is very important, when seeing any claim about “totally positive” matrices, to ascertain which sense of “totally positive” is being used! (This is especially important because many theorems in this subject require the *strict* concept for their validity: see e.g. Section 2.1 below.)