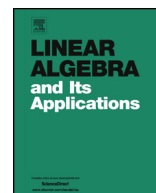




Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa



Minimising the largest mean first passage time of a Markov chain: The influence of directed graphs ☆



Jane Breen^{*}, Steve Kirkland

Department of Mathematics, University of Manitoba, Winnipeg MB, R3T 2N2, Canada

ARTICLE INFO

Article history:

Received 30 November 2016

Accepted 26 January 2017

Available online 30 January 2017

Submitted by R. Brualdi

MSC:

15B51

05C50

60J10

Keywords:

Markov chain

Mean first passage time

Stochastic matrix

Directed graph

Hessenberg matrix

ABSTRACT

For a Markov chain described by an irreducible stochastic matrix A of order n , the mean first passage time $m_{i,j}$ measures the expected time for the Markov chain to reach state j for the first time given that the system begins in state i , thus quantifying the short-term behaviour of the chain. In this article, a lower bound for the maximum mean first passage time is found in terms of the stationary distribution vector of A , and some matrices for which equality is attained are produced. The main objective of this article is to characterise the directed graphs for which any stochastic matrix A respecting this directed graph attains equality in this lower bound, producing a class of Markov chains with optimal short-term behaviour.

© 2017 Published by Elsevier Inc.

☆ Funding: This work was supported in part by NSERC grant no. RGPIN-2014-06123 and by the University of Manitoba Graduate Fellowship.

* Corresponding author.

E-mail addresses: breenj3@myumanitoba.ca (J. Breen), Stephen.Kirkland@umanitoba.ca (S. Kirkland).

1. Introduction

A stochastic matrix A is an entrywise nonnegative matrix whose rows sum to 1, i.e. $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ represents the vector of all ones. Stochastic matrices are central to the study of Markov chains, which are a type of probabilistic model describing dynamical systems which move between some finite number of states in discrete time-steps, where these transitions between states depend only on the present state occupied by the system. The connection with stochastic matrices is the following: given a Markov chain describing a system with a finite state space indexed by the integers $\{1, 2, \dots, n\}$, construct a matrix such that the (i, j) th entry is the probability of the system transitioning from state i to state j in a single time step. This is a stochastic matrix, referred to as the *probability transition matrix*, or simply *transition matrix* of the chain. This transition matrix A wholly represents the Markov chain, in that given an initial vector u_0 describing the probabilities that the process is in one of the various states at time 0, the probability distribution across all states after k time-steps is the vector $u_0^\top A^k$, for $k \geq 1$.

Representing a Markov chain by a stochastic matrix in this way enables us to analyse the long-term behaviour of the modelled system using basic techniques from linear algebra. If the transition matrix A is *irreducible* – i.e. for any pair of indices i, j , there exists some $m \in \mathbb{N}$ such that the (i, j) th entry of A^m is positive – then by the Perron–Frobenius theorem, A must have a strictly positive left eigenvector $w = [w_1 \ w_2 \ \dots \ w_n]^\top$ corresponding to the eigenvalue 1. This eigenvector, when normalised so that the entries sum to 1 (thus producing a probability distribution) is referred to as the *stationary distribution vector* of the chain. It is an important quantity for the following reason: in the case that the transition matrix A is also *primitive* – i.e. there exists some $m \in \mathbb{N}$ for which every entry of A^m is positive – the iterates of the chain converge to w^\top independent of the initial distribution. This is powerful in analysing the underlying system, since we can then say that the probability the Markov chain is in the i th state in the long term is the i th entry w_i of this eigenvector w^\top . Thus the long-term behaviour of the modelled system is summarised by a fundamental feature of the corresponding stochastic matrix.

The short-term behaviour of a system modelled by a Markov chain is considered as follows. Define $F_{i,j}$ to be a random variable representing the first passage time from state i to state j ; i.e. the number of time steps elapsed (≥ 1) before the system reaches state j for the first time, given that it began in state i . The expected value of this random variable, then, is a key quantity of interest. It is referred to as the *mean first passage time from i to j* , denoted $m_{i,j}$. In the special case that $i = j$, $m_{i,i}$ is referred to as the *mean first return time to state i* . This facilitates the construction of the *matrix of mean first passage times* $M = [m_{i,j}]$ which is the unique solution (see [15, Section 6.1]) to the equation

$$M = A(M - M_{\text{diag}}) + J, \quad (1.1)$$

Download English Version:

<https://daneshyari.com/en/article/5773120>

Download Persian Version:

<https://daneshyari.com/article/5773120>

[Daneshyari.com](https://daneshyari.com)