Linear Algebra and its Applications 532 (2017) 8–24

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Infinite and finite dimensional generalized Hilbert tensors ☆

Wei Mei [a], Yisheng Song [b,*]

[a] School of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan, 453007, PR China

[b] School of Mathematics and Information Science and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, Xinxiang, Henan, 453007, PR China

ARTICLE INFO

Article history:
Received 16 November 2016
Accepted 11 May 2017
Available online 23 June 2017
Submitted by R. Brualdi

MSC:
47H15
47H12
34B10
47A52
47J10
47H09
15A48
47H07

Keywords:
Generalized Hilbert tensor
Spectral radius
Norm
Upper bound

ABSTRACT

In this paper, we introduce the concept of an m-order n-dimensional generalized Hilbert tensor $\mathcal{H}_n = (\mathcal{H}_{i_1 i_2 \cdots i_m})$,

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots i_m - m + a},$$
$$a \in \mathbb{R} \setminus \mathbb{Z}^-;\ i_1, i_2, \cdots, i_m = 1, 2, \cdots, n,$$

and show that its H-spectral radius and its Z-spectral radius are smaller than or equal to $M(a)n^{m-1}$ and $M(a)n^{\frac{m}{2}}$, respectively, here $M(a)$ is a constant depending on a. Moreover, both infinite and finite dimensional generalized Hilbert tensors are positive definite for $a \geq 1$. For an m-order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\infty$ with $a > 0$, we prove that $\mathcal{H}_\infty$ defines a bounded and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1 < p < \infty)$. Two upper bounds on the norms of corresponding positively homogeneous operators are obtained.

☆ This work was supported by the National Natural Science Foundation of China (Grant Nos. 11571095, 11601134).

* Corresponding author.
E-mail addresses: 1017187432@qq.com (W. Mei), songyisheng@htu.cn (Y. Song).

http://dx.doi.org/10.1016/j.laa.2017.05.052
0024-3795/

1. Introduction

Let $\mathbb{R}$ denote the set of all real numbers, and let $\mathbb{Z}$ be the set of all integers. We write $\mathbb{Z}^-$ for the set of all non-positive integers and $\mathbb{Z}^{--}$ for the set of all negative integers, i.e.,

$$\mathbb{Z}^- = \{k \in \mathbb{Z} : k \leq 0\} \text{ and } \mathbb{Z}^{--} = \{k \in \mathbb{Z} : k < 0\}.$$

An m-order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m} \in \mathbb{R}$, where $i_j \in I_n = \{1, 2, \cdots, n\}$ for $j \in I_m = \{1, 2, \cdots, m\}$. $\mathcal{A}$ is called a symmetric tensor if the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices. Qi [16,17] introduced the concepts of eigenvalues of the higher order symmetric tensors, and showed the existence of the eigenvalues and some applications. The concepts of real eigenvalue was introduced by Lim [14] independently using a variational approach. Subsequently, many mathematicians studied the spectral properties of various structured tensors under different conditions. The spectral properties of nonnegative matrices had been generalized to nonnegative tensors under various conditions by Chang et al. [1,2], He and Huang [8], He [9], He et al. [10], Li et al. [13], Qi [18,20], Song and Qi [22,23], Wang et al. [26], Yang and Yang [28,29] and references therein.

Study on infinite dimensional higher order tensors has just begun, and fewer work can be found on this topic. Song and Qi [21] introduced the concept of infinite dimensional Hilbert tensor and showed that such a Hilbert tensor defines a bounded, continuous and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1 < p < \infty)$ and the norms of corresponding positively homogeneous operators are not larger than $\frac{\pi}{\sqrt{6}}$. They also proved that the spectral radius and E-spectral radius of finite dimensional Hilbert tensor are smaller than or equal to $n^{m-1} \sin \frac{\pi}{n}$ and $n^{\frac{m}{2}} \sin \frac{\pi}{n}$, respectively. Clearly, an m-order n-dimensional Hilbert tensor is a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{nm})$, introduced by Qi [19]. Also see Chen and Qi [3], Xu [27] for more details of Hankel tensors. A Hilbert tensor (hypermatrix) is a natural extension of Hilbert matrix introduced by Hilbert [7]. For more details of Hilbert matrix, see Frazer [5] and Taussky [25] for n-dimensional Hilbert matrix, Choi [4] and Ingham [11] for an infinite dimensional Hilbert matrix, Magnus [15] and Kato [12] for the spectral properties of infinite dimensional Hilbert matrix.

In this paper, we study more general Hilbert tensor, which is referred to as "generalized Hilbert tensor". The entries of an m-order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m - m + a}, \ a \in \mathbb{R} \setminus \mathbb{Z}^-, i_1, i_2, \cdots, i_m \in \mathbb{Z}^{++} = -\mathbb{Z}^{--}. \tag{1.1}$$