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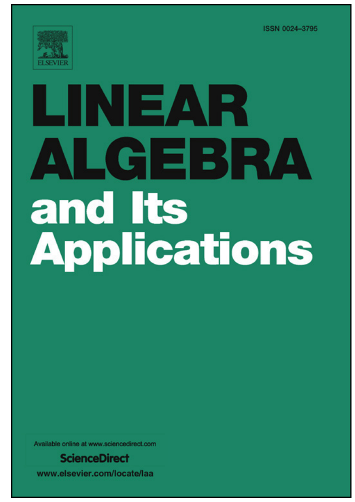

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ON A CONJECTURE OF BHATIA, LIM AND YAMAZAKI

TRUNG HOA DINH, RALUCA DUMITRU, AND JOSE A. FRANCO

ABSTRACT. In this short note we prove a conjecture due to Bhatia, Lim, and Yamazaki on the matrix power means. As a consequence, we obtain a related inequality for the matrix Heron mean and its non-Kubo-Ando extension, conjectured by Bhatia *et. al.*.

1. INTRODUCTION

Let $\mathbb{A}$ denote a set of m positive semidefinite matrices $\{A_1, ..., A_m\}$, and $\{\omega_1, ..., \omega_m\}$ a set of positive weights such that $\sum_i \omega_i = 1$. Bhagwat and Subramanian [1] introduced the power means of $\mathbb{A}$ with weights $\{\omega_1, ..., \omega_m\}$ defined as

$$Q_t(\omega; \mathbb{A}) = \left(\sum_{i=1}^{m} \omega_i A_i^t \right)^{1/t}.$$

These power means are not of Kubo-Ando type [6].

Along different lines, by considering the matrix equation:

$$X = \sum_{i=1}^{m} \omega_i X \#_t A_i,$$

Lim and Pálfia [7] used a fixed point theorem to show that for $0 \leq t \leq 1$ the equation has a unique solution $P_t(\omega; \mathbb{A})$ which is called the power mean of matrices $A_1, A_2, \cdots, A_m$. This result has set off a cascade of articles studying the power means in the literature. One remarkable result is that for $m = 2$, they found a closed form for $P_t(\omega; A, B)$ which is the Kubo-Ando mean corresponding to the representing matrix monotone function $f(\omega, x) = (\omega + (1-\omega)x^t)^{1/t}$ for $-1 \leq t \leq 1$, i.e.,

$$P_t(\omega; A, B) = A^{1/2}\big(\omega I + (1-\omega)(A^{-1/2}BA^{-1/2})^t\big)^{1/t} A^{1/2}.$$

As such, the power means serve as an interpolating family between the arithmetic and geometric means. Indeed, Lim and Pálfia showed that when $\omega_i = 1/m$ for all i's the power means approach the Kracher mean $G(\mathbb{A})$ as $t \to 0$. In particular, for two matrices they obtained

$$\lim_{t \to 0} P_t(\omega; A, B) = A \#_{1-\omega} B. \tag{1}$$

Hence, in the case of two matrices there are two extensions for the power means of numbers: the Kubo-Ando extension $P_t(A, B)$ and the non-Kubo-Ando extension $Q_t(A, B)$.

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