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The number of linear transformations defined on a subspace with given invariant factors

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Given a finite-dimensional vector space V over the finite field $\mathbb{F}_q$ and a subspace W of V, we consider the problem of counting linear transformations $T : W \to V$ which have prescribed invariant factors. The case $W = V$ is a well-studied problem that is essentially equivalent to counting the number of square matrices over $\mathbb{F}_q$ in a conjugacy class and an explicit formula is known in this case. On the other hand, the case of general W is also an interesting problem and there hasn't been substantive progress in this case for over two decades, barring a special case where all the invariant factors of T are of degree zero. We extend this result to the case of arbitrary W by giving an explicit counting formula. As an application of our results, we give new proofs of some recent enumerative results in linear control theory and derive an extension of the Gerstenhaber–Reiner formula for the number of square matrices over $\mathbb{F}_q$ with given characteristic polynomial.

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1. Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements and let $\mathbb{F}_q[x]$ denote the ring of polynomials in the indeterminate x with coefficients in $\mathbb{F}_q$. Throughout this paper, n, k denote positive integers with $n \geq k$. For any ring R, we denote by $M_{n,k}(R)$ the set of all $n \times k$ matrices with entries in R and by $M_n(R)$ the set of all square $n \times n$ matrices with entries in R. Let $\mathrm{GL}_n(\mathbb{F}_q)$ denote the general linear group of nonsingular matrices of $M_n(\mathbb{F}_q)$.

Given any nonzero polynomial matrix $A \in M_{n,k}(\mathbb{F}_q[x])$, there exist invertible matrices $P \in M_n(\mathbb{F}_q[x])$ and $Q \in M_k(\mathbb{F}_q[x])$ such that the product PAQ is of the form

$$\begin{bmatrix} p_1 & 0 & 0 & & \cdots & & 0 \\ 0 & p_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & p_t & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

where all off-diagonal entries are zero and the nonzero entries on the diagonal are monic polynomials $p_i (1 \leq i \leq t)$ satisfying $p_i \mid p_{i+1}$ for $1 \leq i < t$. This diagonal form is known as the Smith normal form of A. We express this concisely by writing $A \sim \mathrm{diag}_{n,k}(p_1, \ldots, p_t)$. The p_i are known as the invariant factors of A and are given by

$$p_i = \frac{\delta_i(A)}{\delta_{i-1}(A)},$$

where $\delta_i(A)$ denotes the ith determinantal divisor of A and equals the greatest common divisor of all $i \times i$ minors of the matrix A.

Let us denote by $I_{n,k}$ the element of $M_{n,k}(\mathbb{F}_q)$ whose (i,j)-th entry is 1 for $i = j$ and zero otherwise. In this paper, we are mainly interested in counting the number of matrices $B \in M_{n,k}(\mathbb{F}_q)$ for which the matrix polynomial $xI_{n,k} - B$ has a prescribed list of invariant factors. More precisely, for any k-tuple of invariant factors, i.e., a tuple $\mathcal{I} = (p_1, \ldots, p_k)$ of monic polynomials in $\mathbb{F}_q[x]$ such that $p_i \mid p_{i+1} (1 \leq i \leq k-1)$, define

$$N_q(n, k; \mathcal{I}) := \# \left\{ B \in M_{n,k}(\mathbb{F}_q) : xI_{n,k} - B \sim \mathrm{diag}_{n,k}(p_1, \ldots, p_k) \right\}.$$

We are interested in a formula for $N_q(n, k; \mathcal{I})$. In fact, the problem of determining $N_q(n, n; \mathcal{I})$ is equivalent to counting the number of square $n \times n$ matrices over $\mathbb{F}_q$ in a conjugacy class. This problem has been studied by Kung [11] and Stong [17] among others, and an explicit formula (see (1)) due to Philip Hall can be found in Stanley [16].

On the other hand, rectangular matrices arise in many contexts in linear control theory and matrix completion problems and are also of considerable interest. We refer to Cravo [1, Thms. 15, 32] for specific examples of matrix completion problems. For