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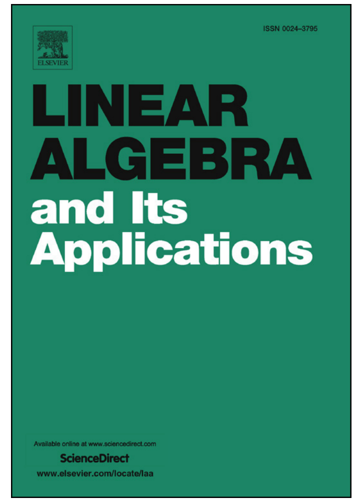

Matrices with multiplicative entries are tensor products

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Matrices with multiplicative entries are tensor products

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Abstract

We study operators which have (infinite) matrix representation whose entries are multiplicative functions of two variables. We show that the such operators are infinite tensor products over the primes. Applications to finding the eigenvalues explicitly of arithmetical matrices are given; also boundedness and norms of Multiplicative Toeplitz and Hankel operators are discussed.

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Introduction

In this paper we shall consider infinite matrices $A = (a_{ij})_{i,j\geq 1}$ whose entries are multiplicative as a function of two variables; i.e. $a_{mn} = f(m, n)$, where $f : \mathbb{N}^2 \to \mathbb{C}$ is not identically zero and satisfies

$$f(m_1n_1, m_2n_2) = f(m_1, m_2)f(n_1, n_2) \quad \text{whenever } (m_1m_2, n_1n_2) = 1.$$

We are interested in knowing when such matrices induce bounded operators (on ℓ^2) and furthermore, what we can say about their (operator) norms and spectra.

The motivation for this investigation is twofold. In a recent paper [10], the singular values (see §1.2 for the definition) of $M_n = M_n(\alpha)$, the $n \times n$ matrix with ij^{th}-entry $(i/j)^{-\alpha}$ if $j|i$ and zero otherwise, were shown to be approximable by the eigenvalues of the operator given by infinite matrix

$$E_\alpha = \Big(\frac{(ij)^\alpha}{[i,j]}\Big)_{i,j\geq 1}.$$

More precisely, with $s_r(M_n)$ denoting the r^{th} largest singular value of M_n and $\lambda_r(E_\alpha)$ the r^{th} largest eigenvalue of E_α, it was shown that for $\alpha < \frac{1}{4}$,

$$s_r(M_n)^2 \sim \lambda_r(E_\alpha)\frac{n^{1-2\alpha}}{1-2\alpha} \quad \text{as } n \to \infty. \tag{1.1}$$

Note that E_α has multiplicative entries. It leads naturally to the question of identifying these eigenvalues and whether (1.1) remains true for $\frac{1}{4} \leq \alpha < \frac{1}{2}$. In particular whether E_α is bounded, indeed compact, for such α — we shall settle the boundedness question here. More generally the above was done for $n \times n$ matrices with entries $f(i/j)$ when $j|i$ and zero otherwise, where f is a square summable multiplicative function on $\mathbb{N}$. See also [14] for related matrices.

Another motivation comes from Multiplicative Toeplitz operators, whose matrix representation has entries of the form $a_{ij} = g(i/j)$ for a given $g : \mathbb{Q}^+ \to \mathbb{C}$. Such operators have been studied in [11], for their connection with Dirichlet series, and in particular the Riemann zeta function. If g is multiplicative as a function on the positive rationals, the matrix has multiplicative entries.

Our main result in this paper is to show that under a natural convergence condition, such matrices A are tensor products of operators over the primes (like an Euler product) with the tensor product corresponding to the prime p having matrix representation $\tilde{A}_p = (f(p^k, p^l))_{k,l\geq 0}$. For finite matrices, this was inspired by a result of Codecá and Nair [4] and generalizes it. The result for infinite matrices can be seen as a limiting case of this.

Thus for example, with $a_{ij} = g(i/j)$ and g multiplicative as above, $\tilde{A}_p$ is the Toeplitz matrix $(g(p^{k-l}))_{k,l\geq 0} = T(a_p)$, where $a_p(t) = \sum_{k=-\infty}^{\infty} g(p^k)t^k$ is the 'symbol'. Then we can deduce