Linear Algebra and its Applications 532 (2017) 231–256

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

James–Stein estimation problem for a multivariate normal random matrix and an improved estimator

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ARTICLE INFO

Article history:
Received 16 November 2016
Accepted 21 June 2017
Available online 27 June 2017
Submitted by W.B. Wu

MSC:
primary 62H12
secondary 62J07, 62C15

Keywords:
Admissibility
James–Stein estimator
James–Stein estimation problem
Lost function
Minimaxity
Multivariate linear model
Shrinkage estimator

ABSTRACT

In this paper, we provide the proof of nonexistence of the James–Stein estimator in the whole parameter space for normal random matrices, equivalently, for multivariate linear regression models, which solves the open problem raised by S.F. Arnold [1]. By introducing the concepts of left and right James–Stein estimators, we obtain the left James–Stein estimator of mean matrix and show that the left James–Stein estimator has minimaxity and optimality in terms of the Efron–Morris type modification. We construct a new minimax combination estimator with lower risk by absorbing the advantages of the left James–Stein estimator and the existing modified Stein estimator. Risk comparisons through finite sample simulation studies illustrate that the proposed combination estimator has a better performance, under the mean-squared error or l_2 risk, compared with all existing estimators.

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http://dx.doi.org/10.1016/j.laa.2017.06.032

1. Introduction

James and Stein [10] developed an important shrinkage estimator, called James–Stein estimator, for a normal random vector by shrinking the ordinary least squares estimator or the conventional estimator. The idea of James–Stein shrinkage dates back to the seminal paper of Stein [15] which showed that the conventional estimator is inadmissible when the dimension of the random vector exceeds two. The James–Stein estimator is a constructive shrinkage estimator which dominates the conventional estimator, and Baranchik [2] shows that positive-part trimming further reduces its risk. For a long time, the conventional estimator was thought to be optimal in every sense and certainly admissible. James and Stein's works corrected the wrong understanding so that their works have had a profound effect on current approaches to estimation in multi-parameter situations, see Chapter 4 of Muirhead [12].

The general version of the estimation problem James and Stein concerned can be stated as follows: for $m \geq 3$, given an m-dimension random vector $\mathbf{z}$ having the $N(\boldsymbol{\mu}, \Sigma)$ distribution with unknown covariance Σ and an $m \times m$ random matrix $\mathbf{S}$ having a Wishart $W_m(v, \Sigma)$ distribution for $v \geq m$ and being independent with $\mathbf{z}$, estimate the unknown mean vector $\boldsymbol{\mu}$. The central goal is to choose a decision vector $\mathbf{d}(\mathbf{z}, \mathbf{S}) = (d_1(\mathbf{z}, \mathbf{S}), \cdots, d_m(\mathbf{z}, \mathbf{S}))'$ from the James–Stein type decision vector set

$$\mathfrak{D}_0 = \left\{\mathbf{d}_c(\mathbf{z}, \mathbf{S}) : \mathbf{d}_c(\mathbf{z}, \mathbf{S}) = \left(1 - \frac{c}{\mathbf{z}'\mathbf{S}\mathbf{z}}\right) \mathbf{z},\ c \geq 0\right\}$$

such that minimizes the mean-squared error or l_2 risk

$$R_0((\boldsymbol{\mu}, \Sigma); \mathbf{d}_c) = \mathbb{E}((\mathbf{d}_c - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{d}_c - \boldsymbol{\mu}))$$

for each decision $\mathbf{d}_c(\mathbf{z}, \mathbf{S})$, where $\mathbb{E}$ is the expectation symbol. Under the l_2 risk R_0, the James–Stein estimator of $\boldsymbol{\mu}$ is

$$\mathbf{d}_{js}(\mathbf{z}, \mathbf{S}) = \left(1 - \frac{m-2}{v-m+3} \frac{1}{\mathbf{z}'\mathbf{S}^{-1}\mathbf{z}}\right) \mathbf{z}. \tag{1.1}$$

If the above James–Stein estimation problem is extended to a normal random matrix $\mathbf{Z}$ with covariance structure of Kronecker product which contains unknown parameters, it is still open. Professor Steven F. Arnold pointed out this open problem early in 1981, see Chapter 19 of Arnold [1].

In this paper, we will investigate the open problem, namely, for $m > 1$ and $p > 1$, given a $p \times m$ random matrix $\mathbf{Z}$ having the $N(\Gamma, I \otimes \Sigma)$ distribution with unknown covariance Σ, where $\otimes$ denotes Kronecker product, and an $m \times m$ random matrix $\mathbf{S}$ having a Wishart $W_m(v, \Sigma)$ distribution for $v \geq m$ and being independent with $\mathbf{Z}$, estimate the unknown mean matrix Γ. Our central aim is to choose a decision matrix $D(\mathbf{Z}, \mathbf{S}) = (d_{ij}(\mathbf{Z}, \mathbf{S}))_{p \times m}$ from the James–Stein type decision matrix set