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Expressing infinite matrices over rings as products of involutions

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ABSTRACT

Let K be an arbitrary field and R be an arbitrary associative ring with identity 1. Słowik in [12] proved that each matrix of $\pm\mathrm{UT}(\infty, K)$ (the group of upper triangular infinite matrices whose entries lying on the main diagonal are equal to either 1 or -1) can be expressed as a product of at most five involutions. In this article, we extend the investigate to an arbitrary associative ring R with identity 1. Our conclusion is that every element of $\pm\mathrm{UT}(\infty, R)$ can be expressed as a product of at most four involutions. We also prove that for the complex field every element of $\Omega\mathrm{T}(\infty, \mathbb{C})$ (the group of upper triangular infinite matrices whose entries lying on the main diagonal satisfy $a\overline{a} = 1$) can be expressed as a product of at most three coninvolutions (matrices satisfying $A\overline{A} = I$).

1. Introduction

Product of involutions is a hot topic investigated by many scholars. Given a group, an involution is an element of order two. Properties of involutions and some other informa-

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tion can be found in [1,9,13]. As a special case, products of two involutions have drawn more attention from scholars. Details can be found in [2–4,7,15]. Based on this, some scholars have turned their attention to the problem of expressing the elements of a group as products of involutions (see [10,11,14,16]). One of the most typical findings (in [5]) is that every $n \times n$ square matrix over a field, with determinant ± 1, is a product of not more than four involutions. Słowik [12] investigated this issue for infinite upper triangular matrices over an arbitrary field. In that case, if a triangular matrix is a product of involutions, its elements from the main diagonal must lie in the set $\{1, -1\}$. Denote the group of elements with such diagonals by $\pm\mathrm{UT}(\infty, K)$. For any field, Słowik proved that every element of this group can be expressed as a product of at most five involutions and for fields of characteristic different from 2, four involutions suffice. In fact, when the characteristic of K is 2, every element g of $\pm\mathrm{UT}(\infty, K)$ can also be expressed as a product of at most four involutions since the decomposition "$g = duv$" in the proof of Theorem 1.1 of [12] is just "$g = uv$" (d is just the identity matrix in that case). In this article, we extend Słowik's result to an arbitrary associative ring with identity 1. What's more, for the complex field $\mathbb{C}$, we prove that every element of $\Omega\mathrm{T}(\infty, \mathbb{C})$ (the group of upper triangular infinite matrices whose entries lying on the main diagonal satisfy $a\overline{a} = 1$) can be expressed as a product of at most three coninvolutions (matrices satisfying $A\overline{A} = I$).

Now we give some notation for this article. Let $\mathrm{T}(n, R)$ and $\mathrm{T}(\infty, R)$ be the group of upper triangular matrices over an associative ring R of dimension n and infinite, respectively. $\mathrm{UT}(n, K)$ and $\mathrm{UT}(\infty, K)$ stand for the group of upper triangular matrices whose entries on the main diagonal are equal to the identity 1. Denote by I_n and I_∞ the $n \times n$ and $\mathbb{N} \times \mathbb{N}$ identity matrices, respectively. An $n \times n$ ($\mathbb{N} \times \mathbb{N}$) matrix A is called an involution if $A^2 = I_n$ ($A^2 = I_\infty$). Denote by a_{ij} the (i, j)-entry of a matrix A. We put

$$\pm\mathrm{UT}(n, R) = \{A \in \mathrm{T}(n, R) | a_{ii} \in \{1, -1\} \text{ for all } 1 \le i \le n\},$$

$$\pm\mathrm{UT}(\infty, R) = \{A \in \mathrm{T}(\infty, R) | a_{ii} \in \{1, -1\} \text{ for all } i \ge 1\},$$

$$\pm\mathrm{D}(n, R) = \{A \in \pm\mathrm{UT}(n, R) | a_{ij} = 0, \text{ for all } i \ne j\},$$

$$\pm\mathrm{D}(\infty, R) = \{A \in \pm\mathrm{UT}(\infty, R) | a_{ij} = 0, \text{ for all } i \ne j\}.$$

The first main result is the following.

Theorem 1.1. *Assume that R is an associative ring with identity 1. Then every element of the group $\pm\mathrm{UT}(n, R)$ and $\pm\mathrm{UT}(\infty, R)$ can be expressed as a product of at most four involutions.*

Define an $n \times n$ ($\mathbb{N} \times \mathbb{N}$) complex matrix A a coninvolution if $A\overline{A} = I_n$ ($A\overline{A} = I_\infty$). Coninvolutions are usually studied in the theory of consimilarity and condiagonalizability (see [6,8]). Let $\mathbb{C}$ be the complex field. We put