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Weyr structures of matrices and relevance to commutative finite-dimensional algebras ☆

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We relate the Weyr structure of a square matrix B to that of the $t \times t$ block upper triangular matrix C that has B down the main diagonal and first superdiagonal, and zeros elsewhere. Of special interest is the case $t = 2$ and where C is the nth Sierpinski matrix B_n, which is defined inductively by $B_0 = 1$ and $B_n = \begin{bmatrix} B_{n-1} & B_{n-1} \\ 0 & B_{n-1} \end{bmatrix}$. This yields an easy derivation of the Weyr structure of B_n as the binomial coefficients arranged in decreasing order. Earlier proofs of the Jordan analogue of this had often relied on deep theorems from such areas as algebraic geometry. The result has interesting consequences for commutative, finite-dimensional algebras.

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Noncommutative finite-dimensional algebras over a field F have been studied almost without pause since the 1840's, with many beautiful results uncovered. A sizable group continues to work on them. But interest in their commutative cousins has only recently been revived. The latter study has been less concerned with the intricacies arising from the particular field F than with the radical. In fact, we will assume F is algebraically closed. Of course, finite-dimensional algebras are finitely generated as algebras. For commutative, finite-dimensional algebras R there are a number of simply-stated, basic problems that remain unanswered. To name one, although we will not pursue this, what is the minimum number of generators of R required in order for some faithful R-module M to have dimension (over F) less than $\dim R$? If R can be generated by k elements, then for $k = 1, 2$, the minimum dimension of a faithful module is $\dim R$. For $k > 3$ there are easy examples where $\dim M < \dim R$. But when $k = 3$ this has been open for over 50 years. The question is better known in the form of whether Gerstenhaber's theorem for two commuting $n \times n$ matrices over F also holds for three: if A, B, C are commuting $n \times n$ matrices, must the dimension of the (unital) subalgebra $F[A, B, C]$ of $M_n(F)$ generated by A, B, C have dimension at most n? (For those interested in further details, such as how algebraic geometry impacts the problem, see [1,2], Chapters 5, 7 of [7], and [4]. This is another instance where the Weyr form seems better suited than its Jordan counterpart.)

The application of our theorems on the Weyr structures of block matrices is to the monomial complete intersection ring

$$B \;=\; F[x_1, x_2, \ldots, x_n]/(x_1^{d_1+1}, \ldots, x_n^{d_n+1})$$

where $d_1, d_2, \ldots, d_n$ are nonnegative integers. (So B is the free commutative algebra on n nilpotent generators with the ith generator having nilpotent index at most $d_i + 1$.) If B_i is the homogeneous space of B of degree i, we give a relatively simple proof of the result that the multiplication map

$$\times(x_1 + x_2 + \cdots + x_n)^{N-2k} : B_k \to B_{N-k}$$

with $N = d_1 + d_2 + \cdots + d_n$ a bijection. This had first been proved by R. Stanley using the hard Lefschetz theorem in algebraic geometry, and later by the second author using the theory of the Lie algebra $sl(2)$. A corollary is that the Weyr structure of the multiplication map $\times(x_1 + x_2 + \cdots + x_n) : B \to B$ is the partition $\dim B = \dim B_0 + \dim B_1 + \cdots + \dim B_N$, once the terms are arranged in decreasing order. As background (we will not pursue this connection), the strong Lefschetz property (which can be defined for an endomorphism of any finite graded vector space) is important when interpreted in terms of a representation of the Lie algebra $sl(2)$. The foundation for these representations, in turn, relies on the Clebsch–Gordan decomposition of modules over $sl(2)$. We mention this in passing because it was discovered by a physicist, and used in quantum mechanics,