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Classification of certain types of maximal matrix subalgebras

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ABSTRACT

Let $\mathcal{M}_n(K)$ denote the algebra of $n \times n$ matrices over a field K of characteristic zero. A nonunital subalgebra $\mathcal{N} \subset \mathcal{M}_n(K)$ will be called a *nonunital intersection* if $\mathcal{N}$ is the intersection of two unital subalgebras of $\mathcal{M}_n(K)$. Appealing to recent work of Agore, we show that for $n \geq 3$, the dimension (over K) of a nonunital intersection is at most $(n-1)(n-2)$, and we completely classify the nonunital intersections of maximum dimension $(n-1)(n-2)$. We also classify the unital subalgebras of maximum dimension properly contained in a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(K)$.

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1. Introduction

Let $\mathcal{M}_n(F)$ denote the algebra of $n \times n$ matrices over a field F. For some interesting sets Λ of subspaces $\mathcal{S} \subset \mathcal{M}_n(F)$, those $\mathcal{S} \in \Lambda$ of maximum dimension over F have been completely classified. For example, a theorem of Gerstenhaber and Serezhkin [7, Theorem 1] states that when Λ is the set of subspaces $\mathcal{S} \subset \mathcal{M}_n(F)$ for which every matrix in $\mathcal{S}$ is nilpotent, then each $\mathcal{S} \in \Lambda$ of maximum dimension is conjugate to the algebra of all strictly upper triangular matrices in $\mathcal{M}_n(F)$. For another example, it is shown in [1, Prop. 2.5] that when Λ is the set of proper unital subalgebras $\mathcal{S} \subset \mathcal{M}_n(F)$ and F is an algebraically closed field of characteristic zero, then each $\mathcal{S} \in \Lambda$ of maximum dimension is a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(F)$.

The goal of this paper is to classify the elements in Λ of maximum dimension in the cases $\Lambda = \Gamma$ and $\Lambda = \Omega$, where the sets Γ and Ω are defined below. First we need some definitions.

Write $\mathcal{M} = \mathcal{M}_n = \mathcal{M}_n(K)$, where K is a field of characteristic zero. (It would be interesting to know if this restriction on the characteristic can be relaxed for the results in this paper.) In the spirit of [3, p. viii], we define a subalgebra of $\mathcal{M}$ to be a vector subspace of $\mathcal{M}$ over K closed under the multiplication of $\mathcal{M}$ (cf. [3, p. 2]); thus a subalgebra need not have a unity, and the unity of a unital subalgebra need not be a unity of the parent algebra. Subalgebras $\mathcal{A}$, $\mathcal{B} \subset \mathcal{M}$ are said to be similar if $\mathcal{A} = \{S^{-1}BS : B \in \mathcal{B}\}$ for some invertible $S \in \mathcal{M}$.

In Isaac's text [4, p. 161], every ring is required to have a unity, but the unity in a subring need not be the same as the unity in its parent ring. Under this definition, a ring may have subrings whose intersection is not a subring. This motivated us to study examples of pairs of unital subalgebras in $\mathcal{M}$ whose intersection $\mathcal{N}$ is nonunital. We call such $\mathcal{N}$ a *nonunital intersection* and we let Γ denote the set of all nonunital intersections $\mathcal{N} \subset \mathcal{M}$. Note that Γ is closed under transposition and conjugation, i.e., if $\mathcal{N} \in \Gamma$, then $\mathcal{N}^{\mathrm{T}} \in \Gamma$ and $S^{-1}\mathcal{N}S \in \Gamma$ for any invertible $S \in \mathcal{M}$.

In order to define Ω, we need to establish additional notation. Let $\mathcal{M}[R_n]$ denote the subalgebra of $\mathcal{M}$ consisting of those matrices whose n-th row is zero. Similarly, $\mathcal{M}[R_n, C_n]$ indicates that the n-th row and n-th column are zero, etc. For $1 \le i, j \le n$, let $E_{i,j}$ denote the elementary matrix in $\mathcal{M}$ with a single entry 1 in row i, column j, and 0 in each of the other $n^2 - 1$ positions. The identity matrix in $\mathcal{M}$ will be denoted by I. For the maximal parabolic subalgebra $\mathcal{P} := \mathcal{M}[R_n] + KE_{n,n}$ in $\mathcal{M}$, define Ω to be the set of proper subalgebras $\mathcal{B}$ of $\mathcal{P}$ with $\mathcal{B} \neq \mathcal{M}[R_n]$.

We now describe Theorems 3.1–3.3, our main results. Theorem 3.1 shows that $\dim \mathcal{N} \le (n-1)(n-2)$ for each $\mathcal{N} \in \Gamma$. Theorem 3.2 shows that up to similarity, $\mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n]$ and $\mathcal{W}^{\mathrm{T}} := \mathcal{M}[R_n, C_{n-1}, C_n]$ are the only subalgebras in Γ having maximum dimension $(n-1)(n-2)$. In Theorem 3.3, we show that $\dim \mathcal{B} \le n^2 - 2n + 3$ for each $\mathcal{B} \in \Omega$, and we classify all $\mathcal{B} \in \Omega$ of maximum dimension $n^2 - 2n + 3$.

The proofs of our theorems depend on four lemmas, which are proved in Section 2. Lemma 2.1 shows that $\mathcal{W}$ (and hence also $\mathcal{W}^{\mathrm{T}}$) is a nonunital intersection of dimension