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Bordering for spectrally arbitrary sign patterns

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We develop a matrix bordering technique that can be applied to an irreducible spectrally arbitrary sign pattern to construct a higher order spectrally arbitrary sign pattern. This technique generalizes a recently developed triangle extension method. We describe recursive constructions of spectrally arbitrary patterns using our bordering technique, and show that a slight variation of this technique can be used to construct inertially arbitrary sign patterns.

1. Introduction

A number of methods have been developed to check that a specific pattern is spectrally or inertially arbitrary, such as the analytic nilpotent-Jacobian method and the algebraic

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nilpotent-centralizer method (see e.g. [4,7–9]), and these have been applied to various classes of patterns (see e.g. [1,5,13]). Recently in [11], a digraph method called *triangle extension* has been developed for constructing higher order spectrally or inertially arbitrary patterns from lower order patterns. In this paper, we generalize the triangle extension method by formulating it as a matrix bordering technique (see Remark 2.2). With this bordering technique, we construct higher order patterns (some of which cannot be obtained by triangle extension) that are spectrally or inertially arbitrary from lower order patterns. We give examples of new spectrally and inertially arbitrary sign patterns obtained by bordering.

1.1. Definitions and the nilpotent-Jacobian method

Given an order n matrix $A = [a_{ij}]$, denote the characteristic polynomial of A by $p_A(z) = \det(zI - A)$. A *sign pattern* is a matrix $\mathcal{A} = [\alpha_{ij}]$ of order n with entries in $\{0, +, -\}$. Let

$$Q(\mathcal{A}) = \{A \mid a_{ij} = 0 \text{ if } \alpha_{ij} = 0, a_{ij} > 0 \text{ if } \alpha_{ij} = + \text{ and } a_{ij} < 0 \text{ if } \alpha_{ij} = -\}.$$

If $A \in Q(\mathcal{A})$ for some pattern $\mathcal{A}$, then A is a *realization* of $\mathcal{A}$ and we sometimes refer to $\mathcal{A}$ as $\operatorname{sgn}(A)$. A pattern $\mathcal{A}$ is *spectrally arbitrary* if for every degree n monic polynomial $p(z)$ over $\mathbb{R}$, there is some real matrix A such that $A \in Q(\mathcal{A})$ and $p_A(z) = p(z)$. A pattern $\mathcal{B} = [\beta_{ij}]$ is a *superpattern* of $\mathcal{A}$ if $\alpha_{ij} \neq 0$ implies $\beta_{ij} = \alpha_{ij}$, and $\mathcal{A}$ is a *subpattern* of $\mathcal{B}$. Two patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{B}$ can be obtained from $\mathcal{A}$ via any combination of negation, transposition, permutation similarity and signature similarity.

A matrix A is *nilpotent* if $A^k = 0$ for some positive integer k and the smallest positive integer k such that $A^k = 0$ is the *index* of A. An order n nilpotent matrix A has characteristic polynomial $p_A(z) = z^n$.

Suppose $\mathcal{A}$ is an order n sign pattern with a nilpotent matrix $A \in Q(\mathcal{A})$ with $m \geq n$ nonzero entries $a_{i_1j_1}, a_{i_2j_2}, \ldots, a_{i_mj_m}$. Let $X = X_A(x_1, x_2, \ldots, x_m)$ denote the matrix obtained from A by replacing $a_{i_kj_k}$ with the variable x_k for $k = 1, \ldots, m$. Writing $p_X(z) = z^n + f_1z^{n-1} + \cdots + f_{n-1}z + f_n$ for some $f_i = f_i(x_1, x_2, \ldots, x_m)$, let $J = J_X$ be the $n \times m$ Jacobian matrix with (i, j) entry equal to $\frac{\partial f_i}{\partial x_j}$ for $1 \leq i \leq n$, and $1 \leq j \leq m$. Let $J_{X=A}$ denote the Jacobian matrix evaluated at the nilpotent realization, that is $J_{X=A} = J|_{(x_1,x_2,\ldots,x_m)=(a_{i_1j_1},a_{i_2j_2},\ldots,a_{i_mj_m})}$. A nilpotent matrix A *allows a full-rank Jacobian* if the rank of $J_{X=A}$ is n. Finding a nilpotent matrix $A \in Q(\mathcal{A})$ that allows a full-rank Jacobian is known as the *nilpotent-Jacobian method*. As noted in part (c) of Theorem 1.1, this method guarantees that every superpattern of $\mathcal{A}$ is spectrally arbitrary.

A matrix A (or pattern $\mathcal{A}$) is *reducible* if there is a permutation matrix P such that PAP^T (resp. $P\mathcal{A}P^T$) is block triangular with more than one nonempty diagonal block. Otherwise it is *irreducible*. A matrix A is *nonderogatory* if the dimension of the eigenspace of every eigenvalue is equal to one. The following theorem combines known results from [4] and [7].