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Some families of operator norm inequalities

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A B S T R A C T

We consider the function $f_{\alpha,\beta}(t) = t^{\gamma(\alpha,\beta)} \prod_{i=1}^{n} \frac{b_i(t^{a_i}-1)}{a_i(t^{b_i}-1)}$ on the interval $(0,\infty)$, where $\alpha = (a_1, a_2, \dots, a_n), \beta = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ and $\gamma(\alpha,\beta) = (1 - \sum_{i=1}^{n}(a_i - b_i))/2$. In [4], Hiai and Kosaki define the relation $\preceq$ using positive definiteness for functions f and g with some suitable conditions and they have proved this relation implies the operator norm inequality associated with functions f and g. In this paper, we give some conditions for $\alpha', \beta' \in \mathbb{R}^m$ to hold the relation $f_{\alpha,\beta}(t) \preceq f_{\alpha',\beta'}(t)$.

1. Introduction

When $f : (0,\infty) \longrightarrow (0,\infty)$ is continuous and satisfies $f(1) = 1$, we denote $f \in C(0,\infty)_1^+$. We call $f \in C(0,\infty)_1^+$ symmetric if it holds $f(t) = tf(1/t)$. For $f, g \in C(0,\infty)_1^+$, we define $f \preceq g$ if the function

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$$\mathbb{R} \ni x \mapsto \frac{f(e^x)}{g(e^x)}$$

is positive definite, where a function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is positive definite means that, for any positive integer n and real numbers $x_1, x_2, \ldots, x_n$, the $n \times n$ matrix $[\varphi(x_i - x_j)]_{i,j=1}^n$ is positive definite, i.e.,

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(x_i - x_j) \geq 0$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. For $f \in C(0,\infty)_1^+$, we define a continuous map $M_f : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ as follows:

$$M_f(s,t) = tf(\frac{s}{t}).$$

Then it holds that $M_f(1,1) = 1$, $M_f(\alpha s, \alpha t) = \alpha M_f(s,t)$ $(\alpha > 0)$ and

$$M_f(s,t) = M_f(t,s)$$

if f is symmetric.

We define the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{M}_N(\mathbb{C})$ by $\langle X, Y \rangle = \mathrm{Tr}(Y^* X)$ for $X, Y \in \mathbb{M}_N(\mathbb{C})$. When $A \in \mathbb{M}_N(\mathbb{C})$, we can define bounded linear operator L_A and R_A on the Hilbert space $(\mathbb{M}_N(\mathbb{C}), \langle \cdot, \cdot \rangle)$ as follows:

$$L_A(X) = AX, \quad R_A(X) = XA \quad \text{ for } X \in \mathbb{M}_N(\mathbb{C}).$$

If both H and K are positive, invertible matrix in $\mathbb{M}_N(\mathbb{C})$ (in short, $H, K > 0$), then L_H and R_K are also positive, invertible operators on $(\mathbb{M}_N(\mathbb{C}), \langle \cdot, \cdot \rangle)$ and satisfy the relation $L_H R_K = R_K L_H$. Using continuous function calculus of operators, we can consider the operator $M_f(L_H, R_K)(= R_K f(L_H R_K^{-1}))$ on $(\mathbb{M}_N(\mathbb{C}), \langle \cdot, \cdot \rangle)$.

In [4], F. Hiai and H. Kosaki has given the following equivalent conditions for $f, g \in C(0,\infty)_1^+$ satisfying the symmetric condition:

(1) there exists a symmetric probability measure ν on $\mathbb{R}$ such that

$$M_f(L_H, R_K)X = \int_{-\infty}^{\infty} H^{is}(M_g(L_H, R_K)X)K^{-is} d\nu(s)$$

for all $H, K, X \in \mathbb{M}_N(\mathbb{C})$ with $H, K > 0$.

(2) $|||M_f(L_H, R_K)X||| \leq |||M_g(L_H, R_K)X|||$ for all $H, K, X \in \mathbb{M}_N(\mathbb{C})$ with $H, K > 0$ and any unitarily invariant norm $||| \cdot |||$, which means $|||UX||| = |||X||| = |||XU|||$ for any unitary $U \in \mathbb{M}_N(\mathbb{C})$ and any matrix $X \in \mathbb{M}_N(\mathbb{C})$.