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On the cardinality of a factor set of binary matrices

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ABSTRACT

The work considers an equivalence relation in the set of all $n \times m$ binary matrices. In each element of the factor-set generated by this relation, we define the concept of canonical binary matrix, namely the minimal element with respect to the lexicographic order. For this purpose, the binary matrices are uniquely represented by ordered n-tuples of integers. We have found a necessary and sufficient condition for an arbitrary matrix to be canonical. This condition could be the base for realizing recursive algorithm for finding all $n \times m$ canonical binary matrices and consequently for finding all bipartite graphs, up to isomorphism with cardinality of each part equal to n and m.

1. Introduction and notation

Let k and n be positive integers, $k \leq n$. By $[n]$ we denote the set

$$[n] = \{1, 2, \ldots, n\}$$

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and by $[k, n]$ the set

$$[k, n] = \{k, k+1, \dots, n\}.$$

A matrix whose elements belong to the set $\mathfrak{B} = \{0, 1\}$ is called *binary* (or *boolean*, or $(0,1)$-*matrix*). With $\mathfrak{B}_{n\times m}$ we will denote the set of all $n \times m$ binary matrices.

By $\mathcal{S}_n$ we denote the symmetric group of order n, i.e. the group of all one-to-one mappings of a set X, $|X| = n$ onto itself. If $x \in X$, $\rho \in \mathcal{S}_n$, then the image of the element x by the mapping ρ is denoted by $\rho(x)$. A square binary matrix is called a *permutation matrix*, if there is exactly one 1 in every row and every column. Let us denote the group of all $n \times n$ permutation matrices by $\mathcal{P}_n$. The isomorphism $\mathcal{P}_n \cong \mathcal{S}_n$ is in effect. It is well known (see [1,2]) that the multiplication of an arbitrary real or complex matrix A from the left with a permutation matrix (if the multiplication is possible) leads to permutation of the rows of the matrix A, while the multiplication of A from the right with a permutation matrix leads to the dislocation of the columns of A.

A *transposition* is a matrix obtained from the identity matrix by interchanging two columns. With $\mathcal{T}_n \subset \mathcal{P}_n$ we denote the set of all transpositions in $\mathcal{P}_n$, i.e. the set of all $n \times n$ permutation matrices, which multiplying from the left an arbitrary $n \times m$ matrix swaps the places of exactly two rows, while multiplying from the right an arbitrary $k \times n$ matrix swaps the places of exactly two columns.

Definition 1. Let $A, B \in \mathfrak{B}_{n\times m}$. We will say that the matrices A and B are *equivalent* and we will write

$$A \sim B,$$

if there exist permutation matrices $X \in \mathcal{P}_n$ and $Y \in \mathcal{P}_m$, such that

$$A = XBY.$$

In other words $A \sim B$, if A is received from B after dislocation of some of the rows and the columns of B. Obviously, the introduced relation is an equivalence relation.

Bipartite graph is the ordered triplet $g = \langle R_g, C_g, E_g\rangle$, where R_g and C_g are non-empty sets such that $R_g \cap C_g = \emptyset$, the elements of which will be called *vertices*. $E_g \subseteq R_g \times C_g = \{\langle r, c\rangle \mid r \in R_g, c \in C_g\}$ – the set of *edges*. Multiple edges are not allowed in our considerations. Let $g' = \langle R_{g'}, C_{g'}, E_{g'}\rangle$ and $g'' = \langle R_{g''}, C_{g''}, E_{g''}\rangle$ be two bipartite graphs. We will say that the graphs g' and g'' are *isomorphic* and we will write $g' \cong g''$, if $|R_{g'}| = |R_{g''}| = n$, $|C_{g'}| = |C_{g''}| = m$ and there exist $\rho \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_m$, such that $\langle r, c\rangle \in E_{g'} \Leftrightarrow \langle \rho(r), \sigma(c)\rangle \in E_{g''}$. For more details on graph theory and its applications, see [3,4].

The connection between the bipartite graphs and the popular puzzle Sudoku is described in details in [5]. There it is shown that if we want to find the number of all $n^2 \times n^2$