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Topological classification of systems of bilinear and sesquilinear forms

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A B S T R A C T

Let $\mathcal{A}$ and $\mathcal{B}$ be two systems consisting of the same vector spaces $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$ and bilinear or sesquilinear forms $A_i, B_i : \mathbb{C}^{n_{k(i)}} \times \mathbb{C}^{n_{l(i)}} \to \mathbb{C}$, for $i = 1, \dots, s$. We prove that $\mathcal{A}$ is transformed to $\mathcal{B}$ by homeomorphisms within $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$ if and only if $\mathcal{A}$ is transformed to $\mathcal{B}$ by linear bijections within $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$.

1. Introduction

Two bilinear or sesquilinear forms $A, B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ are *topologically equivalent* if there exists a homeomorphism (i.e., a continuous bijection whose inverse is also a continuous bijection) $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $A(u, v) = B(\varphi u, \varphi v)$ for all $u, v \in \mathbb{C}^n$. We

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prove that *two forms are topologically equivalent if and only if they are equivalent*; we extend this statement to arbitrary systems of forms. Therefore, the canonical matrices of bilinear and sesquilinear forms given in [10] are also their canonical matrices with respect to topological equivalence.

Two linear operators $A, B : \mathbb{C}^n \to \mathbb{C}^n$ are *topologically equivalent* if there exists a homeomorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $B(\varphi u) = \varphi(Au)$, for all $u \in \mathbb{C}^n$. The problem of classifying linear operators up to topological equivalence is still open. It is solved by Kuiper and Robbin [11,13] for linear operators without eigenvalues that are roots of unity. It is studied for arbitrary operators in [1–3,8,9] and other papers. The fact that the problem of topological classification of forms is incomparably simpler is very unexpected for the authors; three of them only reduce it to the nonsingular case in [5].

Hans Schneider [15] studies the topological space H_r^n of $n \times n$ Hermitian matrices of rank r and proves that its connected components coincide with the *congruence classes. The closure graphs for the congruence classes of 2×2 and 3×3 matrices and for the *congruence classes of 2×2 matrices are constructed in [4,6]. The problems of topological classification of matrix pencils and chains of linear mappings are studied in [7,14].

2. Form representations of mixed graphs

Let G be a *mixed graph*; that is, a graph that may have undirected and directed edges (multiple edges and loops are allowed); let $1, \dots, t$ be its vertices. Its *form representation* $\mathcal{A}$ of dimension $\underline{n} = (n_1, \dots, n_t)$ is given by assigning to each vertex i the vector space $\mathbb{C}^{n_i} := \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (n_i times), to each undirected edge $\alpha : i \,—\, j$ ($i \le j$) a bilinear form $A_\alpha : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \to \mathbb{C}$, and to each directed edge $\beta : i \longrightarrow j$ a sesquilinear form $A_\beta : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \to \mathbb{C}$ that is linear in the first argument and semilinear (i.e., conjugate linear) in the second. Two form representations $\mathcal{A}$ and $\mathcal{B}$ of dimensions $\underline{n}$ and $\underline{m}$ are *topologically isomorphic* (respectively, *linearly isomorphic*) if there exists a family of homeomorphisms (respectively, linear bijections)

$$\varphi_1 : \mathbb{C}^{n_1} \to \mathbb{C}^{m_1}, \ \dots, \ \varphi_t : \mathbb{C}^{n_t} \to \mathbb{C}^{m_t} \tag{1}$$

that transforms $\mathcal{A}$ to $\mathcal{B}$; that is,

$$A_\alpha(u, v) = B_\alpha(\varphi_i u, \varphi_j v), \qquad u \in \mathbb{C}^{n_i}, \ v \in \mathbb{C}^{n_j} \tag{2}$$

for each edge $\alpha : i \,—\, j$ or $i \longrightarrow j$.

Example. *A form representation*

$$\mathcal{A} : \qquad A_\alpha \circlearrowleft \mathbb{C}^{n_1} \underset{A_\gamma}{\overset{A_\beta}{\leftleftarrows}} \mathbb{C}^{n_2} \circlearrowright A_\delta \tag{3}$$