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Note on hook representations of the symmetric group

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ABSTRACT

The (complex) irreducible representations of S_n parametrized by hook shapes are well known to be the exterior powers of the standard representation. We write down a simple explicit expression for the Gelfand–Tsetlin vectors of these representations.

1. Introduction

The (complex) irreducible representations of the symmetric group S_n are parametrized by Young diagrams with n boxes. Given a Young diagram λ with n boxes, a basic problem is to intuitively understand the irreducible S_n-module V^λ parametrized by λ. One way to answer this is to have a combinatorial model of V^λ and then to use the combinatorics of this model to explicitly write down the canonically defined Gelfand–Tsetlin basis (defined below) of V^λ. Let us make this precise.

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Let $\mathcal{Y}$ denote the set of all Young diagrams (there is a unique Young diagram with zero boxes) and $\mathcal{Y}_n$ denote the set of all Young diagrams with n boxes (we write $|\lambda|$ for the number of boxes in a Young diagram λ). A fundamental result (see [1,3,5]) in the representation theory of the symmetric groups constructs, for every positive integer n, a bijection, denoted $\lambda \mapsto V^\lambda$, between Young diagrams with n boxes and equivalence classes of irreducible S_n-modules (we also let V^λ denote an irreducible S_n-module in the corresponding equivalence class) such that properties (i) and (ii) below are satisfied:

(i) *Initialization:* $V^{(2)}$ is the trivial representation of S_2 and $V^{(1,1)}$ is the sign representation of S_2 (here (2), respectively $(1,1)$, denotes the Young diagram with a single row of two boxes, respectively a single column of two boxes).

(ii) *Branching rule:* Given $\mu \in \mathcal{Y}$, we denote by μ^- the set of all Young diagrams obtained from μ by removing one of the inner corners in the Young diagram μ. For $n \geq 2$, given $\lambda \in \mathcal{Y}_n$, consider the irreducible S_n-module V^λ. Treating V^λ as an S_{n-1}-module (we view S_{n-1} as the subgroup of S_n fixing n) we have an S_{n-1}-module isomorphism

$$V^\lambda \cong \bigoplus_{\mu \in \lambda^-} V^\mu.$$

It is a consequence of properties (i) and (ii) above that, for any n, the Young diagram consisting of a single row of n boxes (respectively, a single column of n boxes) corresponds to the trivial representation of S_n (respectively, the sign representation of S_n).

Consider an irreducible S_n-module V^λ, for $\lambda \in \mathcal{Y}_n$. Since the branching is multiplicity free, the decomposition into irreducible S_{n-1}-modules of V^λ is canonical. Each of these modules, in turn, decompose canonically into irreducible S_{n-2}-modules. Iterating this construction, we get a canonical decomposition of V^λ into irreducible S_1-modules, i.e., one dimensional subspaces. Thus, there is a canonical basis of V^λ, determined up to scalars, and called the *Gelfand–Tsetlin (or GZ-) basis* of V^λ.

We now recall the spectral characterization of the GZ-basis [1,4,5].

Let $\mathbb{C}[S_n]$ denote the group algebra of S_n. For $i = 1, 2, \ldots, n$ define $X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}[S_n]$. The X_i's are called the *Young–Jucys–Murphy elements (YJM-elements).* Note that $X_1 = 0$.

Consider the Fourier transform, i.e., the algebra isomorphism

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \in \mathcal{Y}_n} \mathrm{End}(V^\lambda), \tag{1}$$

given by

$$\pi \mapsto (V^\lambda \xrightarrow{\pi} V^\lambda \ : \ \lambda \in \mathcal{Y}_n), \ \ \pi \in S_n.$$

We have identified a canonical basis, the GZ-basis, in each S_n-irreducible. Let $\mathrm{D}(V^\lambda)$ consist of all operators on V^λ diagonal in the GZ-basis of V^λ. It is known that the