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Sign properties of Metzler matrices with applications

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ABSTRACT

Several results about sign properties of Metzler matrices are obtained. It is first established that checking the sign-stability of a Metzler sign-matrix can be either characterized in terms of the Hurwitz stability of the unit sign-matrix in the corresponding qualitative class, or in terms the negativity of the diagonal elements of the Metzler sign-matrix and the acyclicity of the associated directed graph. Similar results are obtained for the case of Metzler block-matrices and Metzler mixed-matrices, the latter being a class of Metzler matrices containing both sign- and real-type entries. The problem of assessing the sign-stability of the convex hull of a finite and summable family of Metzler matrices is also solved, and a necessary and sufficient condition for the existence of common Lyapunov functions for all the matrices in the convex hull is obtained. The concept of sign-stability is then generalized to the concept of $\mathrm{Ker}_+(B)$-sign-stability, a problem that arises in the analysis of certain jump Markov processes. A sufficient condition for the $\mathrm{Ker}_+(B)$-sign-stability of Metzler sign-matrices is obtained and formulated using inverses of sign-matrices and the concept of L^+-matrices. Several applications of the results are discussed in the last section.

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1. Introduction

Metzler matrices [1] are often encountered in various fields such as economics [2] and biology [3–6]. These matrices have also been shown to play a fundamental role in the description of linear positive systems, a class of systems that leave the nonnegative orthant invariant [7]. Due to their particular structure, such systems have been the topic of a lot of works in the dynamical systems and control communities; see e.g. also [8–14] and references therein. In this paper, we will be interested in the sign-properties of such matrices, that is, those general properties that can be deduced from the knowledge of the sign-pattern. This problem initially emerged from economics [15], but also found applications in ecology [16–19] and chemistry [20,21]. The rationale for this approach stems from the fact that, in these fields, the interactions between different participants in a given system are, in general, qualitatively known but quantitatively unknown. This incomplete knowledge is very often a direct consequence of the difficulty in identifying and discriminating models because of their inherent complexity and scarce experimental data. In this context, it seems relevant to study the properties of the system solely based on the sign pattern structure or, more loosely, from the pattern of nonzero entries. This is referred to as *qualitative analysis* [22]. For instance, the sign-stability of general matrices (the property that all the matrices having the same sign-pattern are all Hurwitz stable) has been studied in [15,19,23]. An algorithm, having worst-case $O(n^2)$ time- and space-complexity, verifying the conditions in [23] has been proposed in [24]. Many other problems have also been addressed over the past decades; see e.g. [25–29] and references therein.

The first problem addressed in this paper is the sign-stability problem for which several general necessary and sufficient conditions exist [22]. By adapting these results, we show that the sign-stability of a Metzler sign-matrix can be assessed from the Hurwitz stability of a single particular matrix, referred to as the *unit sign-matrix*, lying inside the qualitative class. Therefore, for this class of matrices, checking the Hurwitz stability of an uncountably infinite and unbounded family of matrices is not more difficult than checking the stability of a given matrix. Lyapunov conditions, taking in this case the form of linear programs [30], can hence be used for establishing the sign-stability of a given Metzler sign-matrix. An alternative condition is formulated in terms of the acyclicity of the graph associated with the sign-pattern, a property that can be easily checked using algorithms such as the Depth-First-Search algorithm [31]. This result is then generalized to the case of block matrices for which several results, potentially enabling the use of distributed algorithms, are provided. Sign-stability results are then generalized to the problem of establishing the sign-stability of all the matrices located in the convex hull of a finite number of “summable” Metzler sign-matrices. This problem is highly connected to the analysis of linear positive switched systems, a problem that has been extensively considered in the literature: see e.g. [32–34]. Necessary and sufficient conditions for the existence of a common Lyapunov function for all the matrices in the convex-hull of such matrices are also provided. A novel stability concept, referred to as