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The Hilton–Milner theorem for the distance-regular graphs of bilinear forms

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ABSTRACT

Let V be an $(n+l)$-dimensional vector space over the finite field $\mathbb{F}_q$ with $l \geq n > 0$, and W be a fixed l-dimensional subspace of V. Suppose $\mathcal{F}$ is a non-trivial intersecting family of n-dimensional subspaces U of V with $U \cap W = 0$. In this paper, we give the tight upper bound for the size of $\mathcal{F}$, and describe the structure of $\mathcal{F}$ which reaches the upper bound.

1. Introduction

Let X be an n-element set and $\binom{X}{k}$ denote the family of all k-subsets of X. A family $\mathcal{F} \subseteq \binom{X}{k}$ is called *intersecting* if for all $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \neq \emptyset$. For any family $\mathcal{F} \subseteq \binom{X}{k}$, the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. We say that $\mathcal{F}$ is *trivial* if $\tau(\mathcal{F}) = 1$. Erdős, Ko and Rado [3] determined the maximum

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size of an intersecting family and showed that any intersecting family with maximum size is trivial.

In 1967, Hilton and Milner [7] determined the maximum size of a non-trivial intersecting family. In 1986, Frankl and Füredi [4] gave a new proof using the shifting technique.

Theorem 1.1. ([7]) *Let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $|X| = n$, $k \geq 2$, $n \geq 2k+1$ and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if*

(i) $\mathcal{F} = \{G \in \binom{X}{k} : x \in G,\ F \cap G \neq \emptyset\} \cup \{F\}$ *for some k-subset F and $x \in X \backslash F$.*
(ii) $\mathcal{F} = \{F \in \binom{X}{3} : |F \cap S| \geq 2\}$ *for some 3-subset S if $k = 3$.*

This theorem is usually called the Hilton–Milner theorem now. In the language of graphs, the Hilton–Milner theorem gives the upper bound on the sizes of subsets of vertices whose maximum distance is $k-1$ and covering number is at least 2 in the Johnson graph $J(n,k)$. Over the years, there have been many interesting extensions of this theorem. See [1] for vector spaces, [11] for set partitions, [12] for weak compositions and so on.

Let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$ and $\begin{bmatrix} V \\ k \end{bmatrix}_q$ denote the family of all k-subspaces of V. For $n, k \in \mathbb{Z}^+$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{0 \leq i < k} \frac{q^{n-i} - 1}{q^{k-i} - 1}.$$

Note that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From now on, we will omit the subscript q.

For two subspaces $A, B \subseteq V$, we say A *intersects* B if $\dim(A \cap B) \geq 1$. A family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is called *intersecting* if A intersects B for all $A, B \in \mathcal{F}$. For any $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, the *covering number* $\tau(\mathcal{F})$ is the minimum dimension of a subspace of V that intersects every element of $\mathcal{F}$. We say that $\mathcal{F}$ is *trivial* if $\tau(\mathcal{F}) = 1$. In [5,6,8,9], using different techniques, the authors determined the maximum size of an intersecting family and showed that any intersecting family with maximum size is trivial. Blokhuis et al. [1] determined the maximum size of a non-trivial intersecting family.

Theorem 1.2. ([1]) *Let $k \geq 3$, and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. For any intersecting family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with $\tau(\mathcal{F}) \geq 2$, we have $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k$. Equality holds only if*

(i) $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : E \subseteq F,\ \dim(F \cap U) \geq 1\} \cup \begin{bmatrix} E+U \\ k \end{bmatrix}$ *for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ with $E \nsubseteq U$.*
(ii) $\mathcal{F} = \{F \in \begin{bmatrix} V \\ 3 \end{bmatrix} : \dim(F \cap S) \geq 2\}$ *for some $S \in \begin{bmatrix} V \\ 3 \end{bmatrix}$ if $k = 3$.*