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Continuity of seminorms on finite-dimensional vector spaces

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ABSTRACT

The main purpose of this note is to establish the continuity of seminorms on finite-dimensional vector spaces over the real or complex numbers.

Throughout this note let $\mathbf{V}$ be a finite-dimensional vector space over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. Let N be a *norm* on $\mathbf{V}$, so that for all $a, b \in \mathbf{V}$ and $\alpha \in \mathbb{F}$,

$$N(a) > 0, \quad a \neq 0,$$

$$N(\alpha a) = |\alpha| N(a),$$

$$N(a+b) \leq N(a) + N(b).$$

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Since $\mathbf{V}$ is finite-dimensional, all norms on $\mathbf{V}$ are equivalent, inducing on $\mathbf{V}$ a unique topology. In particular, it follows that all norms on $\mathbf{V}$ are continuous with respect to this topology.

The notion of norm can be relaxed in two familiar ways:

(a) We say that a real-valued function $f : \mathbf{V} \to \mathbb{R}$ is a *subnorm* on $\mathbf{V}$ if for all $a \in \mathbf{V}$ and $\alpha \in \mathbb{F}$,

$$f(a) > 0, \quad a \neq 0,$$
$$f(\alpha a) = |\alpha| f(a).$$

(b) We say that a real-valued function $S : \mathbf{V} \to \mathbb{R}$ is a *seminorm* on $\mathbf{V}$ if for all $a, b \in \mathbf{V}$ and $\alpha \in \mathbb{F}$,

$$S(a) \geq 0,$$
$$S(\alpha a) = |\alpha| S(a),$$
$$S(a + b) \leq S(a) + S(b).$$

It follows that a subnorm f is a norm on $\mathbf{V}$ if and only if f is *subadditive*. Similarly, a seminorm S is a norm on $\mathbf{V}$ if and only if S is *positive-definite*.

If $\dim \mathbf{V} = 1$, then fixing a nonzero element $a_0 \in \mathbf{V}$, we may write

$$\mathbf{V} = \{\alpha a_0 : \alpha \in \mathbb{F}\}.$$

So every subnorm f on $\mathbf{V}$ must be of the form

$$f(a) = \gamma |\alpha|, \quad a = \alpha a_0 \in \mathbf{V},$$

where γ, the value of f at a_0, is a positive constant and, consequently, f is continuous on $\mathbf{V}$.

If, however, $\dim \mathbf{V} \geq 2$, then contrary to norms, subnorms on $\mathbf{V}$ may fail to be continuous. For example, [1, Section 3], let f be a continuous subnorm on $\mathbf{V}$. Fix an element $a_0 \neq 0$ in $\mathbf{V}$, and let

$$\mathbf{W} = \{\alpha a_0 : \alpha \in \mathbb{F}\}$$

be the one-dimensional linear subspace of $\mathbf{V}$ generated by a_0. Select a real number κ, $\kappa > 1$, and set

$$g_\kappa = \begin{cases} \kappa f(x), & a \in \mathbf{W}, \\ f(x), & a \in \mathbf{V} \setminus \mathbf{W}. \end{cases}$$