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On bipartite distance-regular graphs with exactly two irreducible T-modules with endpoint two

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ABSTRACT

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let X denote the vertex set of Γ, and let A denote the adjacency matrix of Γ. For $x \in X$ let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, $E_0^*, E_1^*, \ldots, E_D^*$, where for $0 \leq i \leq D$, E_i^* represents the projection onto the ith subconstituent of Γ with respect to x. We refer to T as the *Terwilliger algebra* of Γ with respect to x. An irreducible T-module W is said to be *thin* whenever $\dim E_i^*W \leq 1$ for $0 \leq i \leq D$. By the *endpoint* of W we mean $\min\{i | E_i^*W \neq 0\}$. For $0 \leq i \leq D$, let $\Gamma_i(z)$ denote the set of vertices in X that are distance i from vertex z. Define a parameter Δ_2 in terms of the intersection numbers by $\Delta_2 = (k-2)(c_3-1) - (c_2-1)p^2_{22}$. In this paper we prove the following are equivalent: (i) $\Delta_2 > 0$ and for $2 \leq i \leq D-2$ there exist complex scalars α_i, β_i with the following property: for all $x, y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$ we have $\alpha_i + \beta_i|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|$; (ii) For all $x \in X$ there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra $T(x)$ with endpoint two, and these modules are thin.

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1. Introduction

In this paper we obtain a combinatorial characterization of bipartite distance-regular graphs with exactly two irreducible modules of the Terwilliger algebra of endpoint 2, both of which are thin (see Section 2 for formal definitions). Our combinatorial characterization is closely related to the 2-homogeneous property of Curtin [3] and Nomura [14].

Throughout this introduction let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let X denote the vertex set of Γ. For $x \in X$, let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. It is known that there exists a unique irreducible T-module with endpoint 0, and this module is thin [8, Proposition 8.4]. Moreover, Curtin showed that up to isomorphism Γ has exactly one irreducible T-module with endpoint 1, and this module is thin [4, Corollary 7.7].

We now discuss the irreducible T-modules of endpoint 2. For $0 \leq i \leq D$, let $\Gamma_i(z)$ denote the set of vertices in X that are distance i from vertex z. In [7, Theorem 3.11], Curtin proved that the following are equivalent: (i) For all i $(2 \leq i \leq D-2)$ and for all $x, y, z \in X$ with $\partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i$, the number $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ is independent of x, y, z; (ii) For all $x \in X$ there exists a unique irreducible T-module for the Terwilliger algebra $T(x)$ with endpoint 2, and this module is thin. When these equivalent conditions hold, Γ is said to be *almost* 2-*homogeneous*.

Now define a parameter Δ_2 in terms of the intersection numbers by $\Delta_2 = (k-2)(c_3 - 1) - (c_2 - 1)p^2_{22}$. In this paper we prove the following are equivalent: (i) $\Delta_2 > 0$ and for $2 \leq i \leq D-2$ there exist complex scalars α_i, β_i with the following property: for all $x, y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$ we have $\alpha_i + \beta_i|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|$; (ii) For all $x \in X$ there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra $T(x)$ with endpoint two, and these modules are thin. We also compute α_i, β_i in terms of the intersection numbers of Γ.

We remark that this paper is part of a continuing effort to understand and classify the bipartite distance-regular graphs with at most two irreducible modules of the Terwilliger algebra with endpoint 2, both of which are thin. Please see [5–7,10–12] for more work from this ongoing project.

2. Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are

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