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The effect of edge weights on clique weights

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ABSTRACT

Suppose the edges of the complete r-graph on n vertices are weighted with real values. For $r \le k \le n$, the weight of a k-clique is the sum of the weights of its edges. Given the largest gap between the weights of two distinct edges, how small can the largest gap between the weights of two distinct k-cliques be? We answer this question precisely.

1. Introduction

All hypergraphs considered in this paper are uniform, i.e. they are r-graphs for some $r \ge 2$. The complete r-graph on n vertices is denoted by K_n^r. For an r-graph H, let $E(H)$ denote its edge set and $V(H)$ denote its vertex set. We assume that $V(K_n^r) = [n]$. Let $\binom{[n]}{k}$ denote the set of k-cliques of K_n^r where $r \le k \le n$. We are interested in weighings of the edges of K_n^r and their effect on the weights of larger cliques in K_n^r.

A *weighing* of K_n^r is a function $w : \binom{[n]}{r} \to \mathbb{R}$. Observe that any weighing of K_n^r induces a weighing of its subgraphs, where the weight of a subgraph is the sum of the weights of its edges. Trivially, if w is constant, then the weight of any two subgraphs

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with the same number of edges is the same. Now suppose that w is far from constant, what can be said about the weights of all subgraphs with the same number of edges and how far are they from being constant? In particular, what can be said about the weights of the k-cliques? We state this basic question more formally as follows.

Given $w : \binom{[n]}{r} \to \mathbb{R}$, and $r \le k \le n$, let

$$disc_k(w) = \max_{A,B \in \binom{[n]}{k}} |w(A) - w(B)| \,.$$

Notice that $disc(w) = disc_r(w)$ is just the maximum discrepancy between the weights of any two edges, i.e. the maximum gap between two values of w. The extremal question which emerges is to determine:

$$disc(r,k,n) = \min_{w} \frac{disc_k(w)}{disc(w)}$$

where $r < k \le n$ and the minimum is taken over all non-constant weighings w of K_n^r.

Our main motivation for this question (besides being natural on its own right) is that it is closely related to *inclusion matrices* and their generalized inverses. Inclusion matrices (see Section 2 for a definition) have been introduced by Gottlieb [3] and have since been well studied, mainly with respect to their rank, with applications in several areas such as quasi-randomness, see [1,2,4,5,8]. Our approach computes their generalized inverse, which, as it turns out, gives additional information and in particular assists in determining $disc(r,k,n)$.

One can easily determine $disc(r,k,n)$ when $k > n-r$. Indeed, trivially, $disc_n(w) = 0$ so $disc(r,n,n) = 0$. More generally, if $k > n-r$, then the number of elements in $\binom{[n]}{k}$ is smaller than the number of edges so the system of linear equations indexed by $\binom{[n]}{k}$ where each equation is just the sum of all variables corresponding to the edges contained in the k-set corresponding to that equation, has a nontrivial solution. Namely, we can have all weights of k-cliques 0 while w is not constant. Thus, $disc(r,k,n) = 0$ for $k > n-r$. This ceases to be the case when $k \le n-r$. Our main result determines $disc(r,k,n)$ for all relevant values of k.

To state our result, define for $0 \le t \le r < k$:

$$q(t,r,k) = (-1)^t \frac{\binom{k-r+t-1}{t}}{\binom{r}{t}\binom{k}{r}} \,.$$

Theorem 1. *For integers $2 \le r < k \le n-r$ we have $disc(r,k,n) = disc(r,k,k+r)$ and furthermore,*

$$disc(r,k,n) = \frac{2}{\max_{s=0}^{r}\left(\sum_{x=0}^{r}\sum_{y=0}^{r}\left(\sum_{j=0}^{\min\{x,y\}}\binom{s}{j}\binom{r-s}{x-j}\binom{r-s}{y-j}\binom{k-r+s}{r-y-x+j}\right)|q(x,r,k)-q(y,r,k)|\right)} \,.$$