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Hypergraphs and hypermatrices with symmetric spectrum

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ABSTRACT

It is well known that a graph is bipartite if and only if the spectrum of its adjacency matrix is symmetric. In the present paper, this assertion is dissected into three separate matrix results of wider scope, which are extended to hypermatrices. To this end, the concept of bipartiteness is generalized by a new monotone property of cubical hypermatrices, called odd-colorable matrices. It is shown that a nonnegative symmetric r-matrix A has a symmetric spectrum if and only if r is even and A is odd-colorable. This result also solves a problem of Pearson and Zhang about hypergraphs with symmetric spectrum and disproves a conjecture of Zhou, Sun, Wang, and Bu.

Separately, similar results are obtained for the H-spectrum of hypermatrices.

1. Introduction

The purpose of this paper is to extend the following well-known result in spectral graph theory:

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Theorem B. *A graph is bipartite if and only if its adjacency matrix has a symmetric spectrum.*

Recall that the spectrum of a complex square matrix A is called *symmetric* if it is the same as the spectrum of $-A$.

Notwithstanding the fame of Theorem B, we find that it is a certain mismatch, obtained by forcing together several more general statements, with no regard to their key differences. To clarify this point we shall distill a few one-sided implications from the mix of Theorem B.

Thus, for any $n \times n$ complex matrix $A = [a_{i,j}]$ and nonempty sets $I \subset [n]$, $J \subset [n]$, write $A[I, J]$ for the submatrix of all $a_{i,j}$ with $i \in I$ and $j \in J$. Now, call A *bipartite* if there is a partition $[n] = U \cup W$ such that $A[U, U] = 0$ and $A[W, W] = 0$. Clearly, the adjacency matrix of a bipartite graph is bipartite, but the above definition extends to any square matrix.

As with graphs, by negating eigenvectors over one of the partition sets, we get:

Proposition 1. *If a matrix is bipartite, then its spectrum is symmetric.*

Clearly, Proposition 1 immediately implies half of Theorem B, but is much more general, and has nothing to do with graphs. Aiming at the other half of Theorem B, note that the existence of a general converse of Proposition 1 is highly unlikely. Indeed, the matrix

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

dashes hopes for a converse of Proposition 1 even within the class of real symmetric matrices; additionally, the Kronecker powers of H_2 provide infinitely many examples to the same effect.

Furthermore, letting I_n be the identity matrix of order n, we see that the matrices

$$\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

are cospectral; yet the first one is bipartite, whereas the second one is not. Therefore, in general, bipartiteness cannot be inferred from the spectra, even for real symmetric matrices. Obviously, pairing Proposition 1 with a converse in the spirit of Theorem B will badly squash its scope, so it is better left as is. Hints for possible development are given in Proposition 6 and Question 7 below.

Nonetheless, it is interesting to find for which matrices spectral conditions may imply bipartiteness. With this goal in mind, we narrow the focus to nonnegative matrices, and arrive at the following statement: